Fractional Integro - Differentiation As Case of Fox H - transform

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11:00 - 12:00 WeP2L Plenary ICFDA 2024

Abstract

For many years, Oleg has worked with the Meijer G and Fox H functions and has helped to build the largest collection of their particular cases, wherein one can find about 150 named functions and their combinations. Recently, in collaboration with Paco, he has implemented his results in the Wolfram Function Repository with the four functions MeijerGForm, FoxHForm, GenericIntegralTransform, and FractionalOrderD, and has made corresponding talks at the Wolfram' s annual Technology Conference. In this talk, we will describe the structure of majority of the one-dimensional integral transforms (including Riemann - Liouville fractional integro - differentiation as case) in terms of Mellin - Barnes integrals containing Fox H functions in the kernel.

Citations

Biography

https: / / en.wikipedia.org / wiki / Oleg_Marichev

https: / / functions.wolfram.com / About / developers.html

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Main talk

FractionalOrderD: Defining the Riemann-Liouville-Hadamard integro-derivative in the Wolfram language

It is section from "Overview of fractional calculus and its computer implementation in Wolfram Mathematica" by O . I . Marichev, E . L . Shishkina

Definition

We describe the Riemann-Liouville integro-diffirentiation of an arbitrary function to arbitrary symbolic order α in the Wolfram Language. Using techniques described below, this operation, hereafter referred to as "fractional differentiation", has been published in the Wolfram Function Repository as ResourceFunction["FractionalOrderD"]. Defined via an integral transform, fractional differentiation is an analytic function of α which coincides with the usual α-th order derivative when α is a positive integer and with repeated indefinite integration for negative integer α.

We will use notation $\mathcal{D}_z^\alpha[f(z)]$ for Riemann - Liouville integro - diffirentiation for all $\alpha \in \mathbb{C}$.

Definition. By definition of $\mathcal{D}_z^\alpha[f(z)]$ we put

$$\mathcal{D}_z^\alpha[f[z]] ==$$

$$\begin{cases} f[z] & \alpha == 0 \\ f^{(\alpha)}[z] & \alpha \in \mathbb{Z} \,\&\&\, \alpha > 0 \\ \underbrace{\int_0^z dt \ldots \int_0^t dt \int_0^t}_{-\alpha \text{ times}} f[t]\,dt & \alpha \in \mathbb{Z} \,\&\&\, \alpha < 0 \\ \frac{1}{\text{Gamma}[n-\alpha]} D\left[\int_0^z \frac{f[t]}{(z-t)^{\alpha-n+1}} dt, \{z, n\}\right] & n == 1 + \text{Floor}[\text{Re}[\alpha]] \,\&\&\, \text{Re}[\alpha] > 0 \\ \frac{1}{\text{Gamma}[-\alpha]} \int_0^z \frac{f[t]}{(z-t)^{\alpha+1}} dt & \text{Re}[\alpha] < 0 \\ \frac{1}{\text{Gamma}[1-\alpha]} D\left[\int_0^z \frac{f[t]}{(z-t)^{\alpha}} dt, z\right] & \text{Re}[\alpha] == 0 \,\&\&\, \text{Im}[\alpha] \neq 0 \end{cases}$$

where in the case of divergent integral we use Hadamard finite part. Such integro-differentiation is called Riemann-Liouville-Hadamard derivative.

Above we separated cases of symbolic positive integer n-th order derivatives from generic result integro - differentiation of fractional order. In particular for $\alpha = -1, -2, \ldots$ we have equality

$$\frac{1}{\text{Gamma}[-\alpha]} \int_0^z \frac{f[t]}{(z-t)^{\alpha+1}} dt == \underbrace{\int_0^z dt \ldots \int_0^t dt \int_0^t f[t]\,dt}_{-\alpha \text{ times}}$$

So we can combine the third and fifth formulas. Function FractionalOrderD realized regularized Riemann - Liouville integro - diffirentiation. That means if any of the above integrals diverges we use Hadamard regularization of this integral.

Some examples

Let us consider how the function **FractionalOrderD** acts to simple functions. For example, ResourceFunction["FractionalOrderD"][x^2, {x, α}] gives

In[∘]:=

```
ResourceFunction["FractionalOrderD"][x^2, {x, α}]
```

Out[∘]=

$$\frac{2\,x^{2-\alpha}}{\text{Gamma}[3-\alpha]}$$

In[∘]:=

```
ResourceFunction["FractionalOrderD"][Sin[z], {z, α}] // Simplify
```

Out[∘]=

$$\begin{cases} \text{Sin}\left[z+\frac{\pi\alpha}{2}\right] & \alpha \in \mathbb{Z}\ \&\&\ \alpha \geq 0 \\ 2^{-1+\alpha}\sqrt{\pi}\, z^{1-\alpha}\, \text{HypergeometricPFQRegularized}\left[\{1\}, \left\{1-\frac{\alpha}{2}, \frac{3-\alpha}{2}\right\}, -\frac{z^2}{4}\right] & \text{True} \end{cases}$$

If we put $\alpha = 3$ in previous result we obtain $-\cos(z)$.

For other example when $f(z) = e^z$ and $\alpha = -3$ we have the following relations (from the lines 3 and 5 of above formula):

$$\mathcal{D}_z^{-3}[f[z]] == \frac{1}{\text{Gamma}[3]} \int_0^z \frac{f[t]}{(z-t)^{-3+1}}\, dt ==$$

$$\int_0^z \left(\int_0^{t_3} \left(\int_0^{t_2} e^{t_1}\, dt_1\right) dt_2\right) dt_3 == e^z - \frac{z^2}{2} - z - 1$$

Above value also follows from evaluation FractionalOrderD:

In[∘]:=

```
ResourceFunction["FractionalOrderD"][ℯ^z, {z, -3}] // FunctionExpand // Simplify
```

Out[∘]=

$$-1 + e^z - z - \frac{z^2}{2}$$

ResourceFunction["FractionalOrderD"][f[z], {z, a}] == $\mathcal{D}_z^{\alpha}[f[z]]$

In the cases $f(z) = e^z$ and $\alpha = -1/2$ or $\alpha = 1/2$ we come to the following results

$$\mathcal{D}_z^{-1/2}[e^z] == \frac{1}{\text{Gamma}\left[\frac{1}{2}\right]} \int_0^z \frac{e^t}{(z-t)^{1/2}} dt == e^z \, \text{Erf}\left[\sqrt{z}\right]$$

$$\mathcal{D}_z^{1/2}[e^z] == \frac{1}{\text{Gamma}\left[\frac{1}{2}\right]} \text{D}\left[\int_0^z \frac{e^t}{(z-t)^{1/2}} dt, z\right] == \frac{\frac{1}{\sqrt{z}} + e^z \sqrt{\pi}\, \text{Erf}\left[\sqrt{z}\right]}{\sqrt{\pi}}$$

where $\text{erf}(z)$ is the integral of the Gaussian distribution, defined by the formula $\text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt$. These results we can also get using FractionalOrderD:

In[ ]:=

```
ResourceFunction["FractionalOrderD"][ⅇ^z, {z, #}] & /@ {-1/2, 1/2} //
  FunctionExpand // Simplify
```

Out[ ]=

$$\left\{e^z \, \text{Erf}\left[\sqrt{z}\right], \frac{1}{\sqrt{\pi}\sqrt{z}} + e^z \, \text{Erf}\left[\sqrt{z}\right]\right\}$$

Currently operator FractionalOrderD was tested with more than 100000 examples of functions, which includes practically all well known named analytical elementary and special functions and their compositions. These functions can be divide on the following groups: the “simplest” mathematical functions, involving only one or two letters $\frac{1}{z}$, $\sqrt{z}$, z^b, a^z, e^z, z^z; the “named functions” with head - titles like log: $\log(z)$, $\cos(z)$, $J_\nu(z)$, $I_z(b)$; the “composed functions” $\sqrt{z^2}$, $(z^a)^b$, a^{z^c}, $\sin^{-1}(z^3)$; the abstract generic functions $f(z)$, $f(z)^{g(z)}$, $f(g(z))$ and others. If we apply usual differentiation or indefinite integration, we find that not each integral and even derivative can be evaluated in existing computer systems. For listed above 14 first functions we do not get values in Mathematica for below three cases:

$$\left\{\frac{\partial I_z(b)}{\partial z}, \int z^z \, dz, \int I_z(b)\, dz\right\}$$

For other mathematical functions we get the following values of derivatives and indefinite integrals, including elementary and special functions

In[∘]:=

```
{Inactive[D][#, z] == D[#, z]} & /@
 {1/z, Sqrt[z], z^b, a^z, E^z, z^z, Log[z], Cos[z], BesselJ[ν, z], Sqrt[z^2], (z^a)^b, a^z^c, ArcSin[z^3]}
```

Out[∘]=

$$\left\{\left\{\partial_z \frac{1}{z} == -\frac{1}{z^2}\right\}, \left\{\partial_z \sqrt{z} == \frac{1}{2\sqrt{z}}\right\}, \{\partial_z z^b == b\, z^{-1+b}\}, \{\partial_z a^z == a^z \operatorname{Log}[a]\},\right.$$
$$\{\partial_z e^z == e^z\}, \{\partial_z z^z == z^z (1 + \operatorname{Log}[z])\}, \left\{\partial_z \operatorname{Log}[z] == \frac{1}{z}\right\}, \{\partial_z \operatorname{Cos}[z] == -\operatorname{Sin}[z]\},$$
$$\left\{\partial_z \operatorname{BesselJ}[\nu, z] == \frac{1}{2} (\operatorname{BesselJ}[-1+\nu, z] - \operatorname{BesselJ}[1+\nu, z])\right\}, \left\{\partial_z \sqrt{z^2} == \frac{z}{\sqrt{z^2}}\right\},$$
$$\left.\{\partial_z (z^a)^b == a\, b\, z^{-1+a} (z^a)^{-1+b}\}, \{\partial_z a^{z^c} == a^{z^c} c\, z^{-1+c} \operatorname{Log}[a]\}, \left\{\partial_z \operatorname{ArcSin}[z^3] == \frac{3 z^2}{\sqrt{1-z^6}}\right\}\right\}$$

In[∘]:=

```
{Inactive[Integrate][#, z] == Integrate[#, z]} & /@
 {1/z, Sqrt[z], z^b, a^z, E^z, z^z, Log[z], Cos[z], BesselJ[ν, z], Sqrt[z^2], (z^a)^b, a^z^c, ArcSin[z^3]}
```

Out[∘]=

$$\left\{\left\{\int \frac{1}{z}\, dz == \operatorname{Log}[z]\right\}, \left\{\int \sqrt{z}\, dz == \frac{2 z^{3/2}}{3}\right\}, \left\{\int z^b\, dz == \frac{z^{1+b}}{1+b}\right\},\right.$$
$$\left\{\int a^z\, dz == \frac{a^z}{\operatorname{Log}[a]}\right\}, \left\{\int e^z\, dz == e^z\right\}, \left\{\int z^z\, dz == \int z^z\, dz\right\},$$
$$\left\{\int \operatorname{Log}[z]\, dz == -z + z \operatorname{Log}[z]\right\}, \left\{\int \operatorname{Cos}[z]\, dz == \operatorname{Sin}[z]\right\}, \left\{\int \operatorname{BesselJ}[\nu, z]\, dz ==\right.$$
$$\left.2^{-1-\nu} z^{1+\nu} \operatorname{Gamma}\left[\frac{1+\nu}{2}\right] \operatorname{HypergeometricPFQRegularized}\left[\left\{\frac{1+\nu}{2}\right\}, \left\{1+\nu, \frac{3+\nu}{2}\right\}, -\frac{z^2}{4}\right]\right\},$$
$$\left\{\int \sqrt{z^2}\, dz == \frac{z\sqrt{z^2}}{2}\right\}, \left\{\int (z^a)^b\, dz == \frac{z (z^a)^b}{1+a b}\right\},$$
$$\left\{\int a^{z^c}\, dz == -\frac{z \operatorname{Gamma}\left[\frac{1}{c}, -z^c \operatorname{Log}[a]\right] (-z^c \operatorname{Log}[a])^{-1/c}}{c}\right\},$$
$$\left.\left\{\int \operatorname{ArcSin}[z^3]\, dz == z \operatorname{ArcSin}[z^3] - \frac{3}{4} z^4 \operatorname{Hypergeometric2F1}\left[\frac{1}{2}, \frac{2}{3}, \frac{5}{3}, z^6\right]\right\}\right\}$$

New operation FractionalOrderD evaluates repeatable α- integer order derivatives and indefinite integrals and extends results for arbitrary complex or real order α. For instance, we can get the following results for mentioned 14 functions:

$$\mathcal{D}_z^{\alpha}[a] == \frac{a\, z^{-\alpha}}{\text{Gamma}[1-\alpha]}$$

$$\mathcal{D}_z^{\alpha}\left[\frac{1}{z}\right] == \begin{cases} (-1)^{\alpha} z^{-1-\alpha}\, \text{Pochhammer}[1,\alpha] & \alpha \in \mathbb{Z}\ \&\&\ -1<\alpha \\ \frac{z^{-1-\alpha}\,(-\text{EulerGamma}+\text{Log}[z]-\text{PolyGamma}[0,-\alpha])}{\text{Gamma}[-\alpha]} & \text{True} \end{cases}$$

$$\mathcal{D}_z^{\alpha}[z^b] ==$$

$$\begin{cases} (-1)^{\alpha} z^{b-\alpha}\, \text{Pochhammer}[-b,\alpha] & \alpha \in \mathbb{Z}\ \&\&\ b \in \mathbb{Z}\ \&\&\ b<0\ \&\&\ b<\alpha \\ \frac{(-1)^{-1+b} z^{b-\alpha}\,(\text{Log}[z]+\text{PolyGamma}[0,-b]-\text{PolyGamma}[0,1+b-\alpha])}{(-1-b)!\ \text{Gamma}[1+b-\alpha]} & b \in \mathbb{Z}\ \&\&\ b<0 \\ \frac{z^{b-\alpha}\, \text{Gamma}[1+b]}{\text{Gamma}[1+b-\alpha]} & \text{True} \end{cases}$$

$$\mathcal{D}_z^{\alpha}[a^z] ==$$

$$\begin{cases} a^z\, \text{Log}[a]^{\alpha} & \alpha \in \mathbb{Z}\ \&\&\ \alpha \geq 0 \\ a^z\,(1-\text{GammaRegularized}[-\alpha, z\,\text{Log}[a]])\ \text{Log}[a]^{\alpha} & \alpha \in \mathbb{Z}\ \&\&\ \alpha<0 \\ a^z z^{-\alpha}\,(1-\text{GammaRegularized}[-\alpha, z\,\text{Log}[a]])\ (z\,\text{Log}[a])^{\alpha} & \text{True} \end{cases}$$

$$\mathcal{D}_z^{\alpha}[e^z] == \begin{cases} e^z & \alpha \in \mathbb{Z}\ \&\&\ \alpha \geq 0 \\ e^z\,(1-\text{GammaRegularized}[-\alpha, z]) & \text{True} \end{cases}$$

$$\mathcal{D}_z^{\alpha}[\text{BesselJ}[v,z]] ==$$

$$2^{\alpha-2v}\sqrt{\pi}\, z^{-\alpha+v}\, \text{Gamma}[1+v]\ \text{HypergeometricPFQRegularized}\left[\left\{\frac{1+v}{2}, \frac{2+v}{2}\right\},\right.$$

$$\left.\left\{\frac{1}{2}(1-\alpha+v), \frac{1}{2}(2-\alpha+v), 1+v\right\}, -\frac{z^2}{4}\right] \quad (*\text{after FullSimplify}*)$$

$$\mathcal{D}_z^{\alpha}\left[\sqrt{z^2}\right] == \frac{z^{-\alpha}\sqrt{z^2}}{\text{Gamma}[2-\alpha]}$$

$$\mathcal{D}_z^{\alpha}[(z^a)^b] == z^{-\alpha}(z^a)^b \begin{cases} (-1)^{\alpha}\, \text{Pochhammer}[-a\,b,\alpha] & \alpha \in \mathbb{Z}\ \&\&\ a\,b \in \\ ((-1)^{-1+a\,b}\,(\text{Log}[z]+\text{PolyGamma}[0,-a\,b]- \text{PolyGamma}[0,1+a\,b-\alpha]))/((-1-a\,b)!\ \text{Gamma}[1+a\,b-\alpha]) & a\,b \in \mathbb{Z}\ \&\&\ a\,b \\ \frac{\text{Gamma}[1+a\,b]}{\text{Gamma}[1+a\,b-\alpha]} & \text{True} \end{cases}$$

$$\mathcal{D}_z^{\alpha}[\mathrm{ArcSin}[z^3]] ==$$
$$2^{\alpha} \times 3^{-\frac{5}{2}+\alpha} \pi^{5/2} z^{3-\alpha}\, \mathrm{HypergeometricPFQRegularized}\left[\left\{\frac{1}{2}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1, \frac{7}{6}, \frac{4}{3}\right\},\right.$$
$$\left.\left\{\frac{2}{3} - \frac{\alpha}{6}, \frac{5}{6} - \frac{\alpha}{6}, 1 - \frac{\alpha}{6}, \frac{7}{6} - \frac{\alpha}{6}, \frac{4}{3} - \frac{\alpha}{6}, \frac{3}{2} - \frac{\alpha}{6}\right\}, z^6\right] \quad (*\text{after FullSimplify}*)$$

We see that some from these 14 functions are the compositions of other functions. By this reason it is important to collect list of named not composed functions with their integro-derivatives and generic rules for integro-differentiation, which will allow operate with this collection. We have made such collection with 465 functions and call it Alphabet465 and generic 24 rules (see below GenList24):

In[∘]:=

```
Alphabet465 = {a^z, ℯ^z, √z, z^b, AiryAi[z], AiryAiPrime[z], AiryBi[z],
   AiryBiPrime[z], AlternatingFactorial[z], AngerJ[a, z], AngerJ[z, a],
   AngerJ[a, b, z], AngerJ[a, z, b], AngerJ[z, a, b], AppellF1[a, b1, b2, c, x, z],
   AppellF1[a, b1, b2, c, z, y], AppellF1[a, b1, b2, z, x, y], AppellF1[a, b1, z, d, x, y],
   AppellF1[a, z, c, d, x, y], AppellF1[z, b1, b2, d, x, y], ArcCos[z], ArcCosh[z],
   ArcCot[z], ArcCoth[z], ArcCsc[z], ArcCsch[z], ArcSec[z], ArcSech[z],
   ArcSin[z], ArcSinh[z], ArcTan[z], ArcTan[a, z], ArcTan[z, a], ArcTanh[z],
   ArithmeticGeometricMean[a, z], ArithmeticGeometricMean[b, z],
   BarnesG[z], BellB[v, z], BernoulliB[a, z], BesselI[a, z], BesselI[z, a],
   BesselJ[a, z], BesselJ[z, a], BesselK[a, z], BesselK[z, a], BesselY[a, z],
   BesselY[z, a], Beta[a, z], Beta[z, b], Beta[a, b, z], Beta[a, z, c],
   Beta[z, a, b], Beta[a, b, c, z], Beta[a, b, z, d], Beta[c, z, a, b],
   Beta[z, c, a, b], BetaRegularized[a, b, z], BetaRegularized[a, z, c],
   BetaRegularized[z, a, b], BetaRegularized[a, b, c, z], BetaRegularized[a, b, z, d],
   BetaRegularized[c, z, a, b], BetaRegularized[z, c, a, b], Binomial[a, z],
   Binomial[z, b], CarlsonRC[x, z1], CarlsonRC[z, y], CarlsonRD[x, y, z1],
   CarlsonRD[x, z1, a], CarlsonRD[y, z1, a], CarlsonRE[x, z1], CarlsonRE[y, z],
   CarlsonRF[a, x, z1], CarlsonRF[a, y, z], CarlsonRF[x, y, z1], CarlsonRG[a, x, z1],
   CarlsonRG[a, y, z], CarlsonRG[x, y, z1], CarlsonRJ[a, x, y, z1],
   CarlsonRJ[a, x, z1, b], CarlsonRJ[a, y, z, b], CarlsonRJ[x, y, z1, b],
   CarlsonRK[x, z1], CarlsonRK[y, z], CarlsonRM[x, y, z1], CarlsonRM[x, z1, a],
   CarlsonRM[y, z, a], CatalanNumber[z], ChebyshevT[a, z],
   ChebyshevT[z, b], ChebyshevU[a, z], ChebyshevU[z, b], Cos[z], Cosh[z],
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CoshIntegral[z], CosIntegral[z], Cot[z], Coth[z], CoulombF[a, b, z],
CoulombG[a, b, z], CoulombH1[a, b, z], CoulombH2[a, b, z], Csc[z],
Csch[z], DawsonF[z], EllipticE[z], EllipticE[a, z], EllipticE[z, b],
EllipticF[a, z], EllipticF[z, b], EllipticK[z], EllipticNomeQ[z],
EllipticPi[a, z], EllipticPi[z, b], EllipticPi[a, b, z], EllipticPi[a, z, b],
EllipticPi[z, b, c], EllipticTheta[2, z], EllipticTheta[3, z], EllipticTheta[4, z],
EllipticTheta[1, z, a], EllipticTheta[1, v, z], EllipticTheta[2, z, a],
EllipticTheta[2, v, z], EllipticTheta[3, z, a], EllipticTheta[3, v, z],
EllipticTheta[4, z, a], EllipticTheta[4, v, z], EllipticThetaPrime[1, z],
EllipticThetaPrime[1, z, a], EllipticThetaPrime[1, v, z],
EllipticThetaPrime[2, z, a], EllipticThetaPrime[2, v, z],
EllipticThetaPrime[3, z, a], EllipticThetaPrime[3, v, z],
EllipticThetaPrime[4, z, a], EllipticThetaPrime[4, v, z], Erf[z],
Erf[a, z], Erf[z, b], Erfc[z], Erfi[z], EulerE[n, z], ExpIntegralE[a, z],
ExpIntegralE[z, b], ExpIntegralEi[z], z!, z!!, FactorialPower[a, z],
FactorialPower[z, b], FactorialPower[a, b, z], FactorialPower[a, z, c],
FactorialPower[z, b, c], Fibonacci[z], Fibonacci[a, z], Fibonacci[z, b],
FoxH[{{{a_1, α_1}}, {{a_2, α_1}}}, {{{b_1, β_1}}, {{b_2, β_2}}}, z],
FoxH[{Table[{a_i, α_i}, {i, 1, n}], Table[{a_i, α_i}, {i, 1 + n, p}]},
  {Table[{b_i, β_i}, {i, 1, m}], Table[{b_i, β_i}, {i, 1 + m, q}]}, z],
FresnelC[z], FresnelF[z], FresnelG[z], FresnelS[z], Gamma[z],
Gamma[a, z], Gamma[z, a], Gamma[a, b, z], Gamma[a, z, b],
Gamma[z, b, c], GammaRegularized[a, z], GammaRegularized[z, b],
GammaRegularized[a, b, z], GammaRegularized[a, z, b],
GammaRegularized[z, b, c], GegenbauerC[a, b, z], GegenbauerC[a, z, b],
GegenbauerC[z, a, b], Gudermannian[z], HankelH1[a, z],
HankelH1[z, a], HankelH2[a, z], HankelH2[z, a], HarmonicNumber[z],
HarmonicNumber[a, z], HarmonicNumber[z, a], Haversine[z], HermiteH[a, z],
HermiteH[z, a], HeunB[q, α, γ, δ, ϵ, z], HeunBPrime[q, α, γ, δ, ϵ, z],
HeunC[q, α, γ, δ, ϵ, z], HeunCPrime[q, α, γ, δ, ϵ, z],
HeunD[q, α, γ, δ, ϵ, z], HeunDPrime[q, α, γ, δ, ϵ, z],
HeunG[a, q, α, β, γ, δ, z], HeunGPrime[a, q, α, β, γ, δ, z],
HeunT[q, α, γ, δ, ϵ, z], HeunTPrime[q, α, γ, δ, ϵ, z],
HurwitzLerchPhi[a, b, z], HurwitzLerchPhi[a, z, b], HurwitzLerchPhi[z, a, b],
HurwitzLerchPhi[z, b, c], HurwitzZeta[a, z], HurwitzZeta[z, a],
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Hyperfactorial[z], Hypergeometric0F1[a, z], Hypergeometric0F1[z, a],
Hypergeometric0F1Regularized[a, z], Hypergeometric0F1Regularized[z, a],
Hypergeometric1F1[a, b, z], Hypergeometric1F1[a, z, b],
Hypergeometric1F1[z, b, c], Hypergeometric1F1Regularized[a, b, z],
Hypergeometric1F1Regularized[a, z, b], Hypergeometric1F1Regularized[z, b, c],
Hypergeometric2F1[a, b, c, z], Hypergeometric2F1[a, b, z, c],
Hypergeometric2F1[a, z, b, c], Hypergeometric2F1Regularized[a, b, c, z],
Hypergeometric2F1Regularized[a, b, z, c],
Hypergeometric2F1Regularized[a, z, b, c], HypergeometricPFQ[{},
  {b1, b2, b3, b4}, z], HypergeometricPFQ[{a1, a2}, {b1, b2, b3}, z],
HypergeometricPFQ[Table[ak, {k, 1, p}], Table[bk, {k, 1, q}], z],
HypergeometricPFQRegularized[{a1}, {b1, b2, b3}, z],
HypergeometricPFQRegularized[Table[ak, {k, 1, p}], Table[bk, {k, 1, q}], z],
HypergeometricU[a, b, z], HypergeometricU[a, z, b], HypergeometricU[z, a, b],
InverseBetaRegularized[z, a, b], InverseBetaRegularized[x, z, a, b],
InverseEllipticNomeQ[z], InverseErf[z], InverseErf[x, z], InverseErfc[z],
InverseGammaRegularized[a, z], InverseGammaRegularized[a, x, z],
InverseGudermannian[z], InverseHaversine[z],
InverseJacobiCD[a, z], InverseJacobiCD[z, m], InverseJacobiCN[a, z],
InverseJacobiCN[z, m], InverseJacobiCS[a, z], InverseJacobiCS[z, m],
InverseJacobiDC[a, z], InverseJacobiDC[z, m], InverseJacobiDN[a, z],
InverseJacobiDN[z, m], InverseJacobiDS[a, z], InverseJacobiDS[z, m],
InverseJacobiNC[a, z], InverseJacobiNC[z, m], InverseJacobiND[a, z],
InverseJacobiND[z, m], InverseJacobiNS[a, z], InverseJacobiNS[z, m],
InverseJacobiSC[a, z], InverseJacobiSC[z, m], InverseJacobiSD[a, z],
InverseJacobiSD[z, m], InverseJacobiSN[a, z], InverseJacobiSN[z, m],
InverseWeierstrassP[z, {g2, g3}], JacobiAmplitude[z, m], JacobiCD[z, m],
JacobiCN[z, m], JacobiCS[z, m], JacobiDC[z, m], JacobiDN[z, m],
JacobiDS[z, m], JacobiEpsilon[z, b], JacobiNC[z, m], JacobiND[z, m],
JacobiNS[z, m], JacobiP[a, b, c, z], JacobiP[a, b, z, c], JacobiP[a, z, b, c],
JacobiP[z, a, b, c], JacobiSC[z, m], JacobiSD[z, m], JacobiSN[z, m],
JacobiZeta[a, z], JacobiZeta[z, b], JacobiZN[z, b], KelvinBei[0, z],
KelvinBei[a, z], KelvinBei[z, a], KelvinBer[0, z], KelvinBer[a, z],
KelvinBer[z, a], KelvinKei[0, z], KelvinKei[a, z], KelvinKer[0, z],
KelvinKer[a, z], KleinInvariantJ[z], LaguerreL[a, z], LaguerreL[z, a],
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LaguerreL[a, b, z], LaguerreL[a, z, b], LaguerreL[z, a, b], LameC[ν, j, z, m],
LameCPrime[ν, j, z, m], LameS[ν, j, z, m], LameSPrime[ν, j, z, m],
LegendreP[z, a], LegendreP[ν, z], LegendreP[a, b, z], LegendreP[a, z, b],
LegendreP[z, a, b], LegendreP[a, b, 2, z], LegendreP[a, b, 3, z],
LegendreP[a, z, 2, b], LegendreP[a, z, 3, b], LegendreP[z, a, 2, b],
LegendreP[z, a, 3, b], LegendreQ[z, a], LegendreQ[ν, z], LegendreQ[a, b, z],
LegendreQ[a, z, b], LegendreQ[z, a, b], LegendreQ[a, b, 2, z],
LegendreQ[a, b, 3, z], LegendreQ[a, z, 2, b], LegendreQ[a, z, 3, b],
LegendreQ[z, a, 2, b], LegendreQ[z, a, 3, b], LerchPhi[a, b, z],
LerchPhi[a, z, b], LerchPhi[z, a, b], LerchPhi[z, b, c], $\frac{\text{Log}[a]}{\text{Log}[z]}$, Log[z],
LogBarnesG[z], LogGamma[z], LogIntegral[z], LogisticSigmoid[z],
LucasL[z], LucasL[a, z], LucasL[z, b], MarcumQ[a, b, z], MarcumQ[a, z, c],
MarcumQ[a, b, c, z], MarcumQ[a, b, z, d], MarcumQ[a, z, c, d],
MathieuC[a, q, z], MathieuCPrime[a, q, z], MathieuS[a, q, z],
MathieuSPrime[a, q, z], MeijerG[{{a_1}, {a_2}}, {{b_1, b_2}, {b_3}}, z],
MeijerG[{{a_1}, {a_2, a_3}}, {{b_1, b_2}, {b_3}}, z],
MeijerG[{{a_1}, {a_2, a_3, a_4, a_5}}, {{b_1, b_2}, {b_3, b_4, b_5}}, z],
MeijerG[{Table[a_i, {i, 1, n}], Table[a_{n+j}, {j, 1, −n + p}]},
{Table[b_k, {k, 1, m}], Table[b_{m+l}, {l, 1, −m + q}]}, z, r],
MittagLefflerE[a, z], MittagLefflerE[z, b], MittagLefflerE[a, b, z],
MittagLefflerE[a, z, b], MittagLefflerE[z, a, b], ModularLambda[z],
Multinomial[a, b, z], Multinomial[a, b, c, z], NevilleThetaC[z, b],
NevilleThetaD[z, b], NevilleThetaN[z, b], NevilleThetaS[z, b],
NorlundB[ν, z], NorlundB[ν, z, b], NorlundB[ν, γ, z], OwenT[a, z],
OwenT[z, a], ParabolicCylinderD[a, z], ParabolicCylinderD[z, a],
Pochhammer[a, z], Pochhammer[z, b], PolyGamma[0, z],
PolyGamma[a, z], PolyLog[2, z], PolyLog[a, z], PolyLog[z, a],
PolyLog[2, b, z], PolyLog[a, b, z], PolyLog[z, a, b], ProductLog[z],
ProductLog[a, z], ProductLog[k, z], QFactorial[z, b], QGamma[z, b],
QHypergeometricPFQ[{a_1, ..., a_r}, {b_1, ..., b_s}, q, z], QPochhammer[z, b],
QPochhammer[z, z], QPolyGamma[0, z, a], QPolyGamma[a, z, a],
QPolyGamma[a, z, c], RamanujanTauL[z], RamanujanTauTheta[z],
RamanujanTauZ[z], RiemannSiegelTheta[z], RiemannSiegelZ[z], ScorerGi[z],
ScorerGiPrime[z], ScorerHi[z], ScorerHiPrime[z], Sec[z], Sech[z], Sin[z],

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Sinc[z], Sinh[z], SinhIntegral[z], SinIntegral[z], SphericalBesselJ[a, z],
SphericalBesselJ[z, a], SphericalBesselY[a, z], SphericalHankelH1[a, z],
SphericalHankelH1[z, a], SphericalHankelH2[a, z], SphericalHankelH2[z, a],
SphericalHarmonicY[a, b, c, z], SphericalHarmonicY[a, b, z, c],
SphericalHarmonicY[a, b, z, d], SpheroidalPS[ν, μ, γ, z],
SpheroidalPS[ν, μ, 2, γ, z], SpheroidalPSPrime[ν, μ, γ, z],
SpheroidalPSPrime[ν, μ, 2, γ, z], SpheroidalQS[ν, μ, γ, z],
SpheroidalQS[ν, μ, 2, γ, z], SpheroidalQSPrime[ν, μ, γ, z],
SpheroidalQSPrime[ν, μ, 2, γ, z], StruveH[z, a], StruveH[ν, z],
StruveL[z, a], StruveL[ν, z], Subfactorial[z], z^(1/3), z^(1/5), z^(1/7),
Tan[z], Tanh[z], WeberE[z, a], WeberE[ν, z], WeberE[a, z, b],
WeberE[z, a, b], WeberE[ν, a, z], WhittakerM[a, b, z], WhittakerM[a, z, b],
WhittakerM[z, a, b], WhittakerW[a, b, z], WhittakerW[a, z, b],
WhittakerW[z, a, b], ZernikeR[a, b, z], Zeta[z], Zeta[a, z], Zeta[z, b],
DirichletBeta[z], DirichletEta[z], DirichletLambda[z], ErlangB[a, z],
ErlangB[z, a], ErlangC[a, z], ErlangC[z, a], GegenbauerC[a, z],
GegenbauerC[z, a], Log[a, z], Log[z, a], Log10[z], Log2[z], RiemannXi[z]};
```

```
{#, ResourceFunction["FractionalOrderD"][#, {z, α}]} & /@
  Table[Alphabet465[[kk]], {kk, 1, 8}] // Quiet // TableForm
```

Out[∘]//TableForm=

a^z	$\begin{cases} a^z\,\text{Log}[a]^{\alpha} & \alpha\in\mathbb{Z}\ \&\&\ \alpha\geq \\ a^z\,(1-\text{GammaRegularized}[-\alpha, z\,\text{Log}[a]])\,\text{Log}[a]^{\alpha} & \alpha\in\mathbb{Z}\ \&\&\ \alpha< \\ a^z z^{-\alpha}\,(1-\text{GammaRegularized}[-\alpha, z\,\text{Log}[a]])\,(z\,\text{Log}[a])^{\alpha} & \text{True}\end{cases}$
e^z	$\begin{cases} e^z & \alpha\in\mathbb{Z}\ \&\&\ \alpha\geq 0 \\ e^z\,(1-\text{GammaRegularized}[-\alpha, z]) & \text{True}\end{cases}$
$\sqrt{z}$	$\frac{\sqrt{\pi}\, z^{\frac{1}{2}-\alpha}}{2\,\text{Gamma}[\frac{3}{2}-\alpha]}$
z^b	$\begin{cases} (-1)^{\alpha} z^{b-\alpha}\,\text{Pochhammer}[-b,\alpha] & \alpha\in\mathbb{Z}\ \&\&\ b\in\mathbb{Z}\ \&\&\ b<0\ \& \\ \frac{(-1)^{-1+b} z^{b-\alpha}\,(\text{Log}[z]+\text{PolyGamma}[0,-b]-\text{PolyGamma}[0,1+b-\alpha])}{(-1-b)!\,\text{Gamma}[1+b-\alpha]} & b\in\mathbb{Z}\ \&\&\ b<0 \\ \frac{z^{b-\alpha}\,\text{Gamma}[1+b]}{\text{Gamma}[1+b-\alpha]} & \text{True}\end{cases}$
$\text{AiryAi}[z]$	$3^{-\frac{4}{3}+\alpha} z^{-\alpha} \begin{cases} (3^{2/3}\,\text{Gamma}[\frac{1}{3}]\,\text{HypergeometricPFQRegularized}[\{\frac{1}{3},1\},\{\frac{1-\alpha}{3},\frac{2-\alpha}{3},1-\frac{\alpha}{3}\} \\ \quad z\,\text{Gamma}[\frac{2}{3}]\,\text{HypergeometricPFQRegularized}[\{\frac{2}{3},1\},\{\frac{2-\alpha}{3},1-\frac{\alpha}{3},\frac{4-\alpha}{3}\} \\ \frac{3^{\frac{1}{6}(-1+2\alpha)}\,\text{MeijerG}[\{\{\frac{1-\alpha}{3},\frac{2-\alpha}{3},-\frac{\alpha}{3}\},\{\}\},\{\{\frac{1-\alpha}{3},-\frac{\alpha}{3}\},\{0,\frac{1}{3},\frac{2}{3}\}\},\frac{z}{3^{2/3}},\frac{1}{3}]}{2\pi}\end{cases}$
$\text{AiryAiPrime}[z]$	$3^{-\frac{8}{3}+\alpha} z^{-\alpha} \begin{cases} (3\times 3^{1/3}\,\text{Gamma}[-\frac{1}{3}] \\ \quad \text{HypergeometricPFQRegularized}[\{\frac{2}{3},1\},\{\frac{1-\alpha}{3},\frac{2-\alpha}{3},1-\frac{\alpha}{3}\},\frac{z^3}{9}]+ \\ \quad z^2\,\text{Gamma}[\frac{1}{3}]\,\text{HypergeometricPFQRegularized}[\{1,\frac{4}{3}\},\{1-\frac{\alpha}{3},\frac{4-\alpha}{3},\frac{5-\alpha}{3} \\ -\frac{3^{\frac{1}{6}(1+2\alpha)}\,\text{MeijerG}[\{\{\frac{1-\alpha}{3},\frac{2-\alpha}{3},-\frac{\alpha}{3}\},\{\}\},\{\{\frac{2-\alpha}{3},-\frac{\alpha}{3}\},\{0,\frac{1}{3},\frac{2}{3}\}\},\frac{z}{3^{2/3}},\frac{1}{3}]}{2\pi}\end{cases}$
$\text{AiryBi}[z]$	$3^{-\frac{5}{6}+\alpha} z^{-\alpha} \begin{cases} (3^{2/3}\,\text{Gamma}[\frac{1}{3}]\,\text{HypergeometricPFQRegularized}[\{\frac{1}{3},1\},\{\frac{1-\alpha}{3},\frac{2-\alpha}{3},1-\frac{\alpha}{3}\} \\ \quad z\,\text{Gamma}[\frac{2}{3}]\,\text{HypergeometricPFQRegularized}[\{\frac{2}{3},1\},\{\frac{2-\alpha}{3},1-\frac{\alpha}{3},\frac{4-\alpha}{3}\} \\ 2\times 3^{\frac{1}{6}(-1+2\alpha)}\,\pi\,\text{MeijerG}[\{\{\frac{1-\alpha}{3},-\frac{\alpha}{3}\},\{\frac{1}{6}(1-2\alpha),\frac{2-\alpha}{3}\}\}, \\ \quad \{\{\frac{1-\alpha}{3},-\frac{\alpha}{3}\},\{0,\frac{1}{3},\frac{2}{3},\frac{1}{6}(1-2\alpha)\}\},\frac{z}{3^{2/3}},\frac{1}{3}]\end{cases}$
$\text{AiryBiPrime}[z]$	$3^{-\frac{13}{6}+\alpha} z^{-\alpha} \begin{cases} (9\times 3^{1/3}\,\text{Gamma}[\frac{2}{3}] \\ \quad \text{HypergeometricPFQRegularized}[\{\frac{2}{3},1\},\{\frac{1-\alpha}{3},\frac{2-\alpha}{3},1-\frac{\alpha}{3}\},\frac{z^3}{9}]+ \\ \quad z^2\,\text{Gamma}[\frac{1}{3}]\,\text{HypergeometricPFQRegularized}[\{1,\frac{4}{3}\},\{1-\frac{\alpha}{3},\frac{4-\alpha}{3},\frac{5-\alpha}{3} \\ -2\times 3^{\frac{1}{6}(1+2\alpha)}\,\pi\,\text{MeijerG}[\{\{\frac{2-\alpha}{3},-\frac{\alpha}{3}\},\{\frac{1}{6}(-1-2\alpha),\frac{1-\alpha}{3}\}\}, \\ \quad \{\{\frac{2-\alpha}{3},-\frac{\alpha}{3}\},\{0,\frac{1}{3},\frac{2}{3},\frac{1}{6}(-1-2\alpha)\}\},\frac{z}{3^{2/3}},\frac{1}{3}]\end{cases}$

It is interesting to compare Mathematica operations FractionalOrderD, D and Integrate looking what they are doing with simple function z^{-2}:

In[∘]:=

```
D[1/z^2, {z, 7}] == ResourceFunction["FractionalOrderD"][1/z^2, {z, 7}] == -40320/z^9
```

Out[∘]=

True

Asymptotics of Gamma[z] $\leftrightarrow z^{c\,z}\, b^z\, z^a$

Asymptotic of Gamma[z] at ∞

In[∘]:=

```
Asymptotic[Gamma[z], z → ComplexInfinity] // Simplify
```

Out[∘]=

$$\begin{cases} e^{\frac{1}{12 z}-z} \sqrt{2\pi}\, z^{-\frac{1}{2}+z} & \text{Arg}[z] < \pi \\ e^{\frac{1}{12 z}-z} \sqrt{\frac{\pi}{2}}\, (-z)^{-\frac{1}{2}+z} \text{Csc}[\pi z] & \text{True} \end{cases}$$

$$\text{Gamma}[z] \propto \begin{cases} \sqrt{\frac{\pi}{2}}\, e^{-z}\, (-z)^{z-\frac{1}{2}} \text{Csc}[\pi z] & \text{Arg}[z] == \pi \\ \sqrt{2\pi}\, e^{-z} z^{z-\frac{1}{2}} & \text{True} \end{cases} \; /; \; (\text{Abs}[z] \to \infty)$$

$$\text{Abs}[\text{Gamma}[x + i\, y]] \propto \sqrt{2\pi}\, \text{Abs}[y]^{x-\frac{1}{2}}\, e^{-\frac{\pi}{2}\text{Abs}[y]} \; /; \; (\text{Abs}[y] \to \infty) \land x \in \text{Reals} \land y \in \text{Reals}$$

$$\frac{\text{Gamma}[z+a]}{\text{Gamma}[z+b]} \propto z^{a-b}\left(1 + \frac{(a-b)\,(a+b-1)}{2 z} + O\left[\frac{1}{z^2}\right]\right) \; /; \; \text{Abs}[\text{Arg}[z+a]] < \pi \land (\text{Abs}[z] \to \infty)$$

$$\prod_{j=1}^{m} \text{Gamma}[b_j + \beta_j z] \propto C\, \text{Gamma}[B + \mathbf{B}\, z]\, w^z \; /;$$

$$\text{Abs}[\text{Arg}[b_j + \beta_j z]] < \pi \land (\text{Abs}[z] \to \infty) \land B == \frac{1-m}{2} + \sum_{j=1}^{m} b_j \land$$

$$\mathbf{B} == \sum_{j=1}^{m} \beta_j \land w == \left(\sum_{j=1}^{m} \beta_j\right)^{-\sum_{j=1}^{m}\beta_j} \prod_{j=1}^{m} \beta_j^{\beta_j} \land C == (2\pi)^{\frac{m-1}{2}} \prod_{j=1}^{m} \beta_j^{b_j - 1/2} \left(\sum_{j=1}^{m} \beta_j\right)^{\frac{m}{2} - \sum_{j=1}^{m} b_j}$$

$$\prod_{i=1}^{n}\text{Gamma}[1-a_i-\alpha_i z] \propto \text{Gamma}[1-A-A z]\, w^z \;/;$$

$$\text{Abs}[\text{Arg}[1-a_i-\alpha_i z]] < \pi \wedge (\text{Abs}[z]\to\infty) \wedge A == \frac{1-n}{2}+\sum_{i=1}^{n} a_i \wedge$$

$$A == \sum_{i=1}^{n}\alpha_i \wedge w == \left(\sum_{i=1}^{n}\alpha_i\right)^{\sum_{i=1}^{n}\alpha_i} \prod_{i=1}^{n}\alpha_i^{-\alpha_i} \wedge C == (2\pi)^{\frac{n-1}{2}} \left(\prod_{i=1}^{n}\alpha_i^{\frac{1}{2}-a_i}\right)\left(\sum_{i=1}^{n}\alpha_i\right)^{\frac{n}{2}-\sum_{i=1}^{n}a_i}$$

$$\frac{\left(\prod_{j=1}^{m}\text{Gamma}[b_j+\beta_j z]\right)\prod_{i=1}^{n}\text{Gamma}[1-a_i-\alpha_i z]\, Z^{-z}}{\left(\prod_{i=n+1}^{p}\text{Gamma}[a_i+\alpha_i z]\right)\prod_{j=m+1}^{q}\text{Gamma}[1-b_j-\beta_j z]} \propto$$

$$C\,\frac{\text{Gamma}[B_1+B_1 z]\,\text{Gamma}[1-A_1-A_1 z]}{\text{Gamma}[A_2+A_2 z]\,\text{Gamma}[1-B_2-B_2 z]}\, W^{-z} \;/;$$

$$(\text{Abs}[z]\to\infty) \wedge B_1 == \frac{1-m}{2}+\sum_{j=1}^{m} b_j \wedge B_1 == \sum_{j=1}^{m}\beta_j \wedge A_1 == \frac{1-n}{2}+\sum_{i=1}^{n} a_i \wedge$$

$$A_1 == \sum_{i=1}^{n}\alpha_i \wedge A_2 == \frac{1-p+n}{2}+\sum_{i=n+1}^{p} a_i \wedge A_2 == \sum_{i=n+1}^{p}\alpha_i \wedge B_2 == \frac{1+m-q}{2}+\sum_{j=m+1}^{q} b_j \wedge$$

$$B_2 == \sum_{j=m+1}^{q}\beta_j \wedge W == \frac{\left(\sum_{j=1}^{m}\beta_j\right)^{\sum_{j=1}^{m}\beta_j}\left(\sum_{j=m+1}^{q}\beta_j\right)^{\sum_{j=m+1}^{q}\beta_j}\prod_{i=1}^{p}\alpha_i^{\alpha_i}}{\left(\sum_{i=1}^{n}\alpha_i\right)^{\sum_{i=1}^{n}\alpha_i}\left(\sum_{i=n+1}^{p}\alpha_i\right)^{\sum_{i=n+1}^{p}\alpha_i}\prod_{j=1}^{q}\beta_j^{\beta_j}}\, Z \wedge$$

$$C == \frac{(2\pi)^{m+n-\frac{p+q}{2}}\left(\sum_{i=1}^{n}\alpha_i\right)^{\frac{n}{2}-\sum_{i=1}^{n}a_i}\left(\sum_{j=1}^{m}\beta_j\right)^{\frac{m}{2}-\sum_{j=1}^{m}b_j}\prod_{j=1}^{q}\beta_j^{b_j-1/2}}{\left(\sum_{i=n+1}^{p}\alpha_i\right)^{\frac{p-n}{2}-\sum_{i=n+1}^{p}a_i}\left(\sum_{j=m+1}^{q}\beta_j\right)^{\frac{m-q}{2}+\sum_{j=m+1}^{q}b_j}\prod_{i=1}^{p}\alpha_i^{a_i-1/2}}$$

$$\frac{\prod_{k=1}^{p}\text{Gamma}[\alpha_k z+a_k]}{\prod_{k=1}^{q}\text{Gamma}[\beta_k z+b_k]} \propto (2\pi)^{\frac{p-q}{2}}\,\frac{\prod_{k=1}^{p}\alpha_k^{a_k-\frac{1}{2}}}{\prod_{k=1}^{q}\beta_k^{b_k-\frac{1}{2}}}\,\frac{\prod_{k=1}^{p}\alpha_k^{\alpha_k z}}{\prod_{k=1}^{q}\beta_k^{\beta_k z}}$$

$$\begin{cases} z^{\theta-\frac{1}{2}} & \kappa == 0 \\ \text{Abs}[\kappa]^{\frac{1}{2}+\kappa z-\theta}\,\text{Gamma}[\kappa z+\text{UnitStep}[\kappa]-\text{Sign}[\kappa]\,\theta]^{-\text{Sign}[\kappa]} & \text{True} \end{cases} \;/;$$

$$\kappa == \sum_{k=1}^{q}\beta_k-\sum_{k=1}^{p}\alpha_k \wedge \theta == \sum_{k=1}^{p}a_k-\sum_{k=1}^{q}b_k+\frac{q-p+1}{2} \wedge \alpha_k > 0 \wedge$$

$$\beta_k > 0 \wedge -\pi < \text{Arg}[z] < \pi \wedge (\text{Abs}[z]\to\infty)$$

Asymptotic of Gamma[z] at poles z==0,-1,-2,… and at regular points z== 1,2,…

$$\text{Gamma}[z] \propto \frac{(-1)^n}{n!\,(z+n)}\,(1+O[z+n])\ /;\ (z\to -n)\land n\in\text{Integers}\land n\geq 0$$

$$\text{Gamma}[z] \propto \frac{(-1)^n}{n!\,(z+n)} + \frac{(-1)^n\,\text{PolyGamma}[n+1]}{n!} + O[z+n]\ /;\ (z\to -n)\land n\in\text{Integers}\land n\geq 0$$

$$\text{Gamma}[-n+\epsilon] \propto \frac{(-1)^n}{n!\,\epsilon}\,(1+O[\epsilon])\ /;\ n\in\text{Integers}\land n\geq 0$$

$$\text{Gamma}[-n+\epsilon] \propto \frac{(-1)^n}{n!\,\epsilon}\left(1+\text{PolyGamma}[n+1]\,\epsilon+O\left[\epsilon^2\right]\right)\ /;\ n\in\text{Integers}\land n\geq 0$$

$$\text{Gamma}[-n+\epsilon] == \sum_{k=0}^{\infty}\left(\sum_{j=0}^{\text{Floor}\left[\frac{k}{2}\right]}\frac{(-1)^{j+n}\,2\left(1-2^{2j-1}\right)\pi^{2j}\,\text{BernoulliB}[2j]}{(2j)!}\left(\frac{\text{KroneckerDelta}[k-2j]}{n!}+\frac{\text{UnitStep}[k-2j-1]}{(k-2j)!\,n!}\,\text{BellY}\left[\text{Table}\left[(-1)^i\,\{i!,1,-\text{PolyGamma}[i-1,n+1]\},\{i,k-2j\}\right]\right]\right)\right)\epsilon^{k-1}$$

$$\text{Gamma}[z+\epsilon] \propto \text{Gamma}[z]\,(1+O[\epsilon])\ /;\ \text{Not}[z\in\text{Integers}\land z\leq 0]$$

$$\text{Gamma}[z+\epsilon] \propto \text{Gamma}[z]\left(1+\text{PolyGamma}[z]\,\epsilon+O\left[\epsilon^2\right]\right)\ /;\ \text{Not}[z\in\text{Integers}\land z\leq 0]$$

$$\text{Gamma}[z+\epsilon] == \text{Gamma}[z]\sum_{k=0}^{\infty}\frac{\text{Gamma}^{(k)}[z]}{\text{Gamma}[z]\,k!}\,\epsilon^k\ /;\ \text{Not}[z\in\text{Integers}\land z\leq 0]$$

Some important integrals including Gamma

$$\text{Gamma}[z] == \int_0^{\infty} t^{z-1}\,e^{-t}\,dt\ /;\ \text{Re}[z]>0$$

$$\frac{1}{\text{Gamma}[z]} == \frac{1}{2\pi i}\,\text{ContourIntegrate}\left[t^{-z}\,e^t,\{t,L\}\right]$$ (* (Hankel's contour integral.)

The path of integration L starts at $-\infty - i\ 0$ on the real axis, goes to $-\epsilon - i\ 0$, circles the origin in the counterclockwise direction with radius ϵ to the point $-\epsilon + i\ 0$, and returns to the point $-\infty + i\ 0$. *)

$$\text{Gamma}[z] == \int_0^\infty \left(e^{-t} - \sum_{k=0}^{n} \frac{(-t)^k}{k!}\right) t^{z-1}\, dt \;/;\; n \in \text{Integers} \land n \geq 0 \land -n-1 < \text{Re}[z] < -n$$

$$\text{Gamma}[z] == \int_1^\infty t^{z-1} e^{-t}\, dt + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\ (z+k)} \;/;\; \text{Not}[z \in \text{Integers} \land z \leq 0]$$

$$\int_0^\infty t^{s-1} e^{-t}\, dt == \text{Gamma}[s] \;/;\; \text{Re}[s] > 0$$

$$\int_0^1 t^{s-1} (1-t)^{\beta-1}\, dt == \text{Beta}[s, \beta] == \frac{\text{Gamma}[s]}{\text{Gamma}[\beta+s]} \text{Gamma}[\beta] \;/;$$
$$\text{Re}[s] > 0 \land \text{Re}[\beta] > 0$$

$$\int_0^\infty t^{s-1} (1+t)^b\, dt == \text{Gamma}[s]\ \text{Gamma}[-b-s] \frac{1}{\text{Gamma}[-b]} \;/;$$
$$\text{Re}[s] > 0\ \&\&\ \text{Re}[s+b] < 0$$

$$\int_{\gamma - i\infty}^{\gamma + i\infty} \text{Gamma}[a+t]\ \text{Gamma}[b-t]\ z^{-t}\, dt == 2\pi i \frac{\text{Gamma}[a+b]\ z^a}{(1+z)^{a+b}} \;/;$$
$$-\text{Re}[a] < \gamma < \text{Re}[b] \land \text{Abs}[\text{Arg}[z]] < \pi$$

$$\int_{-\infty}^{\infty} \text{Gamma}[i\, t + \alpha]\ \text{Gamma}[\beta + i\, t]\ \text{Gamma}[\gamma - i\, t]\ \text{Gamma}[\delta - i\, t]\ z^t\, dt ==$$
$$\frac{2\pi z^{\alpha i}\ \text{Gamma}[\alpha+\gamma]\ \text{Gamma}[\beta+\gamma]\ \text{Gamma}[\alpha+\delta]\ \text{Gamma}[\beta+\delta]}{\text{Gamma}[\alpha+\beta+\gamma+\delta]}$$
$$\text{Hypergeometric2F1}[\alpha+\gamma, \alpha+\delta, \alpha+\beta+\gamma+\delta, 1-z^i] \;/;$$
$$\text{Im}[\alpha+\gamma] > 0 \land \text{Im}[\beta+\gamma] > 0 \land \text{Im}[\alpha+\delta] > 0 \land \text{Im}[\beta+\delta] > 0$$

$$\int_{\gamma - i\infty}^{\gamma + i\infty} \text{Gamma}[a+t]\ \text{Gamma}[b+t]\ \text{Gamma}[c-t]\ \text{Gamma}[d-t]\, dt ==$$
$$2\pi i \frac{\text{Gamma}[a+c]\ \text{Gamma}[a+d]\ \text{Gamma}[b+c]\ \text{Gamma}[b+d]}{\text{Gamma}[a+b+c+d]} \;/;$$
$$-\text{Min}[\text{Re}[a], \text{Re}[b]] < \gamma < \text{Min}[\text{Re}[c], \text{Re}[d]]$$

$$\underset{\substack{x_1, x_2, \ldots, x_n > 0 \\ x_1 + x_2 + \ldots + x_n \leq 1}}{\int \ldots \int} x_1^{\alpha_1 - 1} x_2^{\alpha_2 - 1} \ldots x_n^{\alpha_n - 1}\, dx_1\, dx_2 \ldots dx_n == \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\ldots\Gamma(\alpha_n)}{\Gamma(\alpha_1 + \ldots + \alpha_n + 1)}.$$

Derivatives of Gamma[z]

$$\partial_z \text{Gamma}[z] == \text{Gamma}[z]\ \text{PolyGamma}[z]$$

In[∘]:=

$$\text{FullSimplify}\left[\text{Table}\left[\text{Gamma}^{(n)}[z] == \begin{cases} \text{Gamma}[z] & n == 0 \\ \text{Gamma}[z]\,\text{BellY}[\text{Table}[\{1, \text{PolyGamma}[k, z]\}, \{k, 0, n-1\}]] & \text{True} \end{cases}, \{n, 0, 6\}\right]\right]$$

Out[∘]=

{True, True, True, True, True, True, True}

$$\partial_{\{z,n\}}\left(\frac{1}{\text{Gamma}[z]}\right) == \frac{1}{\text{Gamma}[a]}\Bigg(\text{KroneckerDelta}[n] + \text{UnitStep}[n-1]\ \text{BellY}\left[\text{Table}\left[\left\{\frac{(-1)^j\, j!}{\text{Gamma}[a]^j}, \text{Gamma}[a], \text{PolyGamma}[j-1, a]\right\}, \{j, n\}\right]\right]\Bigg)$$

$$\partial_{\{z,\alpha\}}\text{Gamma}[z] == z^{-1-\alpha}\begin{cases} (-1)^{\alpha}\,\alpha! & \alpha \in \mathbb{Z}\ \&\&\ -1 < \alpha \\ \frac{-\text{EulerGamma}+\text{Log}[z]-\text{PolyGamma}[-\alpha]}{\text{Gamma}[-\alpha]} & \text{True} \end{cases} + \sum_{k=1}^{\infty}\frac{\text{BellY}[\text{Table}[\{1, \text{PolyGamma}[j-1, 1]\}, \{j, 1, k\}]]}{k\,\text{Gamma}[k-\alpha]}\, z^{k-\alpha-1}$$

In[∘]:=

$$\text{Gamma}^{(n)}[z]$$

Out[∘]=

$$\text{Gamma}^{(n)}[z]$$

Reflection formula for Gamma[z]

$$\text{Gamma}[z] == \frac{\pi}{\text{Sin}[\pi z]\ \text{Gamma}[1-z]}$$

$$\text{Gamma}[z]\ \text{Gamma}[1-z] == \frac{\pi}{\text{Sin}[\pi z]}\ /;\ \text{Not}[z \in \text{Integers}]$$

$$\text{Gamma}[z]\ \text{Gamma}[n-z] == \frac{\pi}{\text{Sin}[\pi z]}\ \text{Pochhammer}[1-z, n-1]$$

$$\text{Gamma}\left[z+\frac{1}{2}\right]\text{Gamma}\left[\frac{1}{2}-z\right] == \frac{\pi}{\text{Cos}[\pi z]}$$

Multiple arguments for Gamma[z]

$$\text{Gamma}[2z] == \frac{2^{2z-1}}{\sqrt{\pi}}\,\text{Gamma}[z]\,\text{Gamma}\left[z+\frac{1}{2}\right]$$

$$\text{Gamma}[n z + b] == n^{n z + b - \frac{1}{2}}\,(2\pi)^{\frac{1-n}{2}}\prod_{k=0}^{n-1}\text{Gamma}\left[z+\frac{b+k}{n}\right]\ /;\ n \in \text{Integers} \land n > 0$$

$$\frac{\prod_{k=1}^{m}\text{Gamma}[b_k + B_k z]\ \prod_{k=1}^{n}\text{Gamma}[1 - a_k - A_k z]}{\prod_{k=n+1}^{p}\text{Gamma}[a_k + A_k z]\ \prod_{k=m+1}^{q}\text{Gamma}[1 - b_k - B_k z]} ==$$

$$(2\pi)^{m+n-\frac{p+q}{2}+\frac{1}{2}\left(-\sum_{j=1}^{m}B_j-\sum_{j=1}^{n}A_j+\sum_{j=n+1}^{p}A_j+\sum_{j=m+1}^{q}B_j\right)}\ \frac{\prod_{j=1}^{q}B_j^{b_j-\frac{1}{2}}}{\prod_{j=1}^{p}A_j^{a_j-\frac{1}{2}}}$$

$$\frac{\prod_{j=1}^{m}\prod_{k=0}^{B_j-1}\text{Gamma}\left[z+\frac{b_j+k}{B_j}\right]}{\prod_{j=n+1}^{p}\prod_{k=0}^{A_j-1}\text{Gamma}\left[z+\frac{a_j+k}{A_j}\right]}\ \frac{\prod_{j=1}^{n}\prod_{k=0}^{A_j-1}\text{Gamma}\left[\frac{1-a_j+k}{A_j}-z\right]}{\prod_{j=m+1}^{q}\prod_{k=0}^{B_j-1}\text{Gamma}\left[\frac{1-b_j+k}{B_j}-z\right]}\left(\frac{\prod_{j=1}^{q}B_j^{B_j}}{\prod_{j=1}^{p}A_j^{A_j}}\right)^{z}\ /;$$

$$A_j \in \text{Integers} \land A_j > 0 \land 1 \le j \le p \land B_j \in \text{Integers} \land B_j > 0 \land 1 \le j \le q$$

Ratio of gamma functions

$$\frac{\text{Gamma}[z+1]}{\text{Gamma}[z]} == z$$

$$\frac{\text{Gamma}[z-1]}{\text{Gamma}[z]} == \frac{1}{z-1}$$

$$\frac{\text{Gamma}[z+n]}{\text{Gamma}[z]} == \text{Pochhammer}[z, n]$$

$$\frac{\text{Gamma}[z-n]}{\text{Gamma}[z]} == \frac{(-1)^n}{\text{Pochhammer}[1-z, n]}\ /;\ n \in \text{Integers} \land n \ge 0$$

FoxH & MeijerG functions

$$H_{p,q}^{m,n}\left[z \left| \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right.\right] \wedge G_{p,q}^{m,n}\left[z \left| \begin{array}{c} (a_i)_{1,p} \\ (b_j)_{1,q} \end{array} \right.\right] \leftrightarrow$$

$$\frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \text{GAMMAs}[s]\, z^{s-1}\, ds$$

Definition and asymptotics

Definitions of FoxH and MeijerG functions

Information[FoxH]

Out[∘]=

Symbol

FoxH[{{{a_1, α_1}, …, {a_n, α_n}}, {{a_{n+1}, α_{n+1}}, …, {a_p, α_p}}}, {{{b_1, β_1}, …, {b_m, β_m}}, {{b_{m+1}, β_{m+1}}, …, {b_q, β_q}}}, z] is the Fox H-function $H_{p\,q}^{m\,n}\left(z \left| \begin{array}{c} (a_1, \alpha_1), \ldots, (a_p, \alpha_p) \\ (b_1, \beta_1), \ldots, (b_q, \beta_q) \end{array} \right.\right)$.

Documentation Web »

Attributes {NumericFunction, Protected, ReadProtected}

Full Name System`FoxH

FoxH[{{{a_1, α_1}, …, {a_n, α_n}}, {{a_{n+1}, α_{n+1}}, …, {a_p, α_p}}}, {{{b_1, β_1}, …, {b_m, β_m}}, {{b_{m+1}, β_{m+1}}, …, {b_q, β_q}}}, z] ==
FoxH[{Table[{a_i, α_i}, {i, 1, n}], Table[{a_i, α_i}, {i, n + 1, p}]}, {Table[{b_i, β_i}, {i, 1, m}], Table[{b_i, β_i}, {i, m + 1, q}]}, z] ==

$$\frac{1}{2\pi i}\,\text{ContourIntegrate}\left[\frac{\left(\prod_{j=1}^{m}\text{Gamma}[b_j + \beta_j s]\right)\prod_{i=1}^{n}\text{Gamma}[1 - a_i - \alpha_i s]}{\left(\prod_{i=n+1}^{p}\text{Gamma}[a_i + \alpha_i s]\right)\prod_{j=m+1}^{q}\text{Gamma}[1 - b_j - \beta_j s]}\, z^{-s}, \{s, \mathcal{L}\}\right]$$

/; m ∈ Integers ∧ m ≥ 0 ∧ n ∈ Integers ∧ n ≥ 0 ∧ p ∈ Integers ∧ p ≥ 0 ∧ q ∈ Integers ∧ q ≥ 0 ∧ m ≤ q ∧ n ≤ p ∧ α_i ∈ Reals ∧ α_i > 0 ∧ 1 ≤ i ≤ p ∧ β_j ∈ Reals ∧ β_j > 0 ∧ 1 ≤ j ≤ q

$$H_{p,q}^{m,n}\left(z \left| \begin{array}{c} (a_1, \alpha_1), \ldots, (a_n, \alpha_n), (a_{n+1}, \alpha_{n+1}), \ldots, (a_p, \alpha_p) \\ (b_1, \beta_1), \ldots, (b_m, \beta_m), (b_{m+1}, \beta_{m+1}), \ldots, (b_q, \beta_q) \end{array} \right.\right) = \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\left(\prod_{j=1}^{m}\Gamma(b_j + \beta_j s)\right)\prod_{i=1}^{n}\Gamma(1 - a_i - \alpha_i s)}{\left(\prod_{i=n+1}^{p}\Gamma(a_i + s\,\alpha_i)\right)\prod_{j=m+1}^{q}\Gamma(1 - b_j - \beta_j s)}\, z^{-s}\, ds \;/;$$

$$m \in \mathbb{N} \wedge n \in \mathbb{N} \wedge p \in \mathbb{N} \wedge q \in \mathbb{N} \wedge m \le q \wedge n \le p \wedge \alpha_i \in \mathbb{R} \wedge \alpha_i > 0 \wedge 1 \le i \le p \wedge \beta_j \in \mathbb{R} \wedge \beta_j > 0 \wedge 1 \le j \le q$$

```
FoxH[{{{a1, r}, …, {an, r}}, {{an+1, r}, …, {ap, r}}},
   {{{b1, r}, …, {bm, r}}, {{bm+1, r}, …, {bq, r}}}, z] ==
 1/r MeijerG[{{a1, …, an}, {an+1, …, ap}}, {{b1, …, bm}, {bm+1, …, bq}}, z, r] ==
 1/r 1/(2 π ⅈ) ContourIntegrate[
   ((∏_{k=1}^m Gamma[bk + s]) ∏_{k=1}^n Gamma[1 - ak - s]) /
   ((∏_{k=n+1}^p Gamma[ak + s]) ∏_{k=m+1}^q Gamma[1 - bk - s]) z^(-s/r), {s, ℒ}] /; r > 0
```

$$H_{p,q}^{m,n}\left(z \left| \begin{matrix} \{a_1, r\}, \ldots, \{a_n, r\}, \{a_{n+1}, r\}, \ldots, \{a_p, r\} \\ \{b_1, r\}, \ldots, \{b_m, r\}, \{b_{m+1}, r\}, \ldots, \{b_q, r\} \end{matrix} \right.\right) == \frac{1}{r} G_{p,q}^{m,n}\left(z, r \left| \begin{matrix} a_1, \ldots, a_n, a_{n+1}, \ldots, a_p \\ b_1, \ldots, b_m, b_{m+1}, \ldots, b_q \end{matrix} \right.\right) /; r > 0$$

```
FoxH[{{{a1, 1}, …, {an, 1}}, {{an+1, 1}, …, {ap, 1}}},
   {{{b1, 1}, …, {bm, 1}}, {{bm+1, 1}, …, {bq, 1}}}, z] ==
 MeijerG[{{a1, …, an}, {an+1, …, ap}}, {{b1, …, bm}, {bm+1, …, bq}}, z]
```

$$H_{p,q}^{m,n}\left(z \left| \begin{matrix} \{a_1, 1\}, \ldots, \{a_n, 1\}, \{a_{n+1}, 1\}, \ldots, \{a_p, 1\} \\ \{b_1, 1\}, \ldots, \{b_m, 1\}, \{b_{m+1}, 1\}, \ldots, \{b_q, 1\} \end{matrix} \right.\right) == G_{p,q}^{m,n}\left(z \left| \begin{matrix} a_1, \ldots, a_n, a_{n+1}, \ldots, a_p \\ b_1, \ldots, b_m, b_{m+1}, \ldots, b_q \end{matrix} \right.\right)$$

The infinite contour of integration $\mathcal{L}$ separates the poles of $\Gamma(1 - a_k - \alpha_k\, s)$at $s = \frac{1-a_k+j}{\alpha_k}$, $j \in \mathbb{N}$ from the poles of $\Gamma(b_i + \beta_i\, s)$ at $s = -\frac{b_i+l}{\beta_i}$, $l \in \mathbb{N}$. Such a contour always exists in the cases $\beta_i(1 - a_k + j) \neq -\alpha_k(b_i + l)$.

There are three possibilities for the contour $\mathcal{L}$:

(i) $\mathcal{L}$ runs from γ-$i\infty$ to γ+$i\infty$ (where $\mathrm{Im}(\gamma) = 0$) so that all poles of $\Gamma(b_j + \beta_j\, s)$, $m = 1, \ldots, m$, are to the left, and all the poles of $\Gamma(1 - a_i - \alpha_i\, s)$, $i = 1, \ldots, n$, to the right, of $\mathcal{L}$.
(*This contour can be a straight line $(\gamma - i\infty,\ \gamma + i\infty)$ if $\mathrm{Re}(b_i - a_k) > -1$ (then $-\frac{b_i+l}{\beta_i} < \gamma < 1 - \mathrm{Re}(a_k)$). (In this case the integral converges if $p + q < 2\,(m + n)$, $|\mathrm{Arg}(z)| < \left(m + n - \frac{p+q}{2}\right)\pi$. If $m + n - \frac{p+q}{2} = 0$, then z must be real and positive and additional condition $(q - p)\,\gamma + \mathrm{Re}(\mu) < 0$, $\mu == \sum_{l=1}^{q} b_l - \sum_{k=1}^{p} a_k + \frac{p-q}{2} + 1$, should be added.)*)

(ii) $\mathcal{L}$ is a left loop, starting and ending at -∞ and encircling all poles of $\Gamma(b_j + \beta_j\, s)$, $j = 1, \ldots, m$, once in the positive direction, but none of the poles of $\Gamma(1 - a_i - \alpha_i\, s)$, $i = 1, \ldots, n$.
(*(In this case the integral converges if $q \geq 1$ and either $q > p$ or $q = p$ and $|z| < 1$ or $q = p$ and $|z| == 1$and $m + n - \frac{p+q}{2} \geq 0$ and $\mathrm{Re}(\mu) < 0$.)*)

(iii) $\mathcal{L}$ is a right loop, starting and ending at +∞ and encircling all poles of $\Gamma(1 - a_i - \alpha_i\, s)$, $i = 1, \ldots, n$, once in the negative direction, but none of the poles of $\Gamma(b_j + \beta_j\, s)$,

$j = 1, \ldots, m.$

(*(In this case the integral converges if $p \geq 1$ and either $p > q$ or $p = q$ and $|z| > 1$ or $q = p$ and $|z| == 1$and $m + n - \frac{p+q}{2} \geq 0$ and $\operatorname{Re}(\mu) < 0$.)*)

$$H^{m,n}_{p,q}\left[z \,\middle|\, \begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix}\right] == H^{m,n}_{p,q}\left(z \,\middle|\, \begin{matrix} (a_1, \alpha_1), \ldots, (a_n, \alpha_n), (a_{n+1}, \alpha_{n+1}), \ldots, (a_p, \alpha_p) \\ (b_1, \beta_1), \ldots, (b_m, \beta_m), (b_{m+1}, \beta_{m+1}), \ldots, (b_q, \beta_q) \end{matrix}\right) \leftrightarrow$$

$$\begin{pmatrix} \overbrace{+\ldots+}^{\text{m times}} (b_j, \beta_j) & \overbrace{-\ldots-}^{\text{n times}} (a_j, \alpha_j) \\ \underbrace{+\ldots+}_{\text{p-n times}} (a_j, \alpha_j) & \underbrace{-\ldots-}_{\text{q-m times}} (b_j, \beta_j) \end{pmatrix}$$

FoxH[{Table[{a_j, α_j}, {j, 1, n}], Table[{a_j, α_j}, {j, n + 1, p}]},
{Table[{b_j, β_j}, {j, 1, m}], Table[{b_j, β_j}, {j, m + 1, q}]}, z] ==
FoxH[{{{a_1, α_1}, ..., {a_n, α_n}}, {{a_{n+1}, α_{n+1}}, ..., {a_p, α_p}}},
{{{b_1, β_1}, ..., {b_m, β_m}}, {{b_{m+1}, β_{m+1}}, ..., {b_q, β_q}}}, z] ==

$$\frac{1}{2 \pi i} \int_{\mathcal{L}} \frac{\prod_{j=1}^{m} \text{Gamma}[b_j + \beta_j s] \prod_{j=1}^{n} \text{Gamma}[1 - a_j - \alpha_j s]}{\prod_{j=n+1}^{p} \text{Gamma}[a_j + \alpha_j s] \prod_{j=m+1}^{q} \text{Gamma}[1 - b_j - \beta_j s]} x^{-s} \, ds$$

$$H[x] == H^{m,n}_{p,q}\left[x \,\middle|\, \begin{matrix} (a_j, \alpha_j)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix}\right] == \frac{1}{2 \pi i} \int_{\mathcal{L}} \theta[s] \, x^s \, ds$$

$$\theta[s] == \frac{\prod_{j=1}^{n} \text{Gamma}[1 - a_j + \alpha_j s] \prod_{j=1}^{m} \text{Gamma}[b_j - \beta_j s]}{\prod_{j=n+1}^{p} \text{Gamma}[a_j - \alpha_j s] \prod_{j=m+1}^{q} \text{Gamma}[1 - b_j + \beta_j s]}$$

$$s == \frac{b_h + v}{\beta_h} \;/;\; 1 \leq h \leq m \;\&\&\; v == 0, 1, 2, \ldots$$

$$s == \frac{a_i - \eta - 1}{\alpha_i} \;/;\; 1 \leq i \leq n \;\&\&\; \eta == 0, 1, 2, \ldots$$

$$\beta_h (a_i - \eta - 1) \neq \alpha_i (b_h + v)$$

$$\delta == \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j$$

MeijerG[{Table[a_j, {j, 1, n}], Table[a_j, {j, n + 1, p}]},
{Table[b_j, {j, 1, m}], Table[b_j, {j, m + 1, q}]}, z] ==
FoxH[{{a_1, ..., a_n}, {a_{n+1}, ..., a_p}}, {{b_1, ..., b_m}, {b_{m+1}, ..., b_q}}, z] ==

$$\frac{1}{2 \pi i} \int_{\mathcal{L}} \frac{\prod_{j=1}^{m} \text{Gamma}[b_j + s] \prod_{j=1}^{n} \text{Gamma}[1 - a_j - s]}{\prod_{j=n+1}^{p} \text{Gamma}[a_j + s] \prod_{j=m+1}^{q} \text{Gamma}[1 - b_j - s]} x^{-s} \, ds$$

$$G[x] == G_{p,q}^{m,n}\left[x \,\middle|\, \begin{matrix} (a_j)_{1,p} \\ (b_j)_{1,q} \end{matrix}\right] == G_{p,q}^{m,n}\left(z \,\middle|\, \begin{matrix} a_1, \ldots, a_n, a_{n+1}, \ldots, a_p \\ b_1, \ldots, b_m, b_{m+1}, \ldots, b_q \end{matrix}\right) == \frac{1}{2\pi i}\int_{\mathcal{L}} \theta_G[s]\, x^{-s}\, ds$$

$$\theta_G[s] == \frac{\left(\prod_{j=1}^{m} \mathrm{Gamma}[b_j + s]\right) \prod_{j=1}^{n} \mathrm{Gamma}[1 - a_j - s]}{\left(\prod_{j=n+1}^{p} \mathrm{Gamma}[a_j + s]\right) \prod_{j=m+1}^{q} \mathrm{Gamma}[1 - b_j - s]}$$

Main characteristics of FoxH and MeijerG

$$v == \sum_{j=1}^{A} a_j + \sum_{j=1}^{B} b_j - \sum_{j=1}^{C} c_j - \sum_{j=1}^{D} d_j \quad (*\text{Marichev books 1978}*)$$

$$GG[z] == GG_{C,D}^{A,B}\left[z \,\middle|\, \begin{matrix} (a_j)_{1,A}, (b_j)_{1,B} \\ (c_j)_{1,C}, (d_j)_{1,D} \end{matrix}\right] == GG_{C,D}^{A,B}\left[z \,\middle|\, \begin{matrix} (a_A), (b_B) \\ (c_C), (d_D) \end{matrix}\right] ==$$

$$\frac{1}{2\pi i}\int_{\mathcal{L}==C} \frac{\prod_{j=1}^{A} \mathrm{Gamma}[a_j + s] \prod_{j=1}^{B} \mathrm{Gamma}[b_j - s]}{\prod_{j=1}^{C} \mathrm{Gamma}[c_j + s] \prod_{j=1}^{D} \mathrm{Gamma}[d_j - s]} z^{-s}\, ds$$

$$G[z] == G_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix} (a_p) \\ (b_q) \end{matrix}\right] ==$$

$$\frac{1}{2\pi i}\int_{\mathcal{L}==C} \frac{\prod_{j=1}^{m} \mathrm{Gamma}[b_j + s] \prod_{j=1}^{n} \mathrm{Gamma}[1 - a_j - s]}{\prod_{j=n+1}^{p} \mathrm{Gamma}[a_j + s] \prod_{j=m+1}^{q} \mathrm{Gamma}[1 - b_j - s]} z^{-s}\, ds$$

$$c^* == m + n - \frac{p+q}{2}$$

$$\mu == \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j + \frac{p-q}{2} + 1$$

$$H_{p\,q}^{m\,n}\left(z \,\middle|\, \begin{matrix} (a_1, \alpha_1), \ldots, (a_p, \alpha_p) \\ (b_1, \beta_1), \ldots, (b_q, \beta_q) \end{matrix}\right) == \frac{1}{2\pi i}\int_{\mathcal{L}} \frac{\left(\prod_{j=1}^{m} \Gamma(b_j + \beta_j s)\right) \prod_{i=1}^{n} \Gamma(1 - a_i - \alpha_i s)}{\left(\prod_{i=n+1}^{p} \Gamma(a_i + \alpha_i s)\right) \prod_{j=m+1}^{q} \Gamma(1 - b_j - \beta_j s)} z^{-s}\, ds$$

$$f[z] == \frac{1}{2\pi i}\int_{\mathcal{L}} \frac{\prod_{j=1}^{n} \mathrm{Gamma}[b_j - B_j s] \prod_{j=1}^{m} \mathrm{Gamma}[a_j + A_j s]}{\prod_{j=1}^{q} \mathrm{Gamma}[d_j - D_j s] \prod_{j=1}^{p} \mathrm{Gamma}[c_j + C_j s]} z^{s}\, ds ==$$

$$\frac{1}{2\pi i}\int_{\mathcal{L}} \frac{\prod_{j=1}^{n} \mathrm{Gamma}[b_j + B_j s] \prod_{j=1}^{m} \mathrm{Gamma}[a_j - A_j s]}{\prod_{j=1}^{q} \mathrm{Gamma}[d_j + D_j s] \prod_{j=1}^{p} \mathrm{Gamma}[c_j - C_j s]} z^{-s}\, ds\ /;$$

$$\mathcal{L} == \{\gamma - i\,\infty, \gamma + i\,\infty\} \quad (*\text{Bateman Vol 1,49–50 page}*)$$

$$\alpha == \sum_{j=1}^{m} A_j + \sum_{j=1}^{n} B_j - \sum_{j=1}^{p} C_j - \sum_{j=1}^{q} D_j$$

$$\beta == \sum_{j=1}^{m} A_j - \sum_{j=1}^{n} B_j - \sum_{j=1}^{p} C_j + \sum_{j=1}^{q} D_j$$

$$\lambda == \mathrm{Re}\left[\sum_{j=1}^{m} a_j - \frac{m}{2} + \sum_{j=1}^{n} b_j - \frac{n}{2} - \sum_{j=1}^{p} c_j + \frac{p}{2} - \sum_{j=1}^{q} d_j + \frac{q}{2}\right]$$

$$\rho == \prod_{j=1}^{m} A_j^{A_j} \prod_{j=1}^{n} B_j^{-B_j} \prod_{j=1}^{p} C_j^{-C_j} \prod_{j=1}^{q} D_j^{D_j}$$

$$\text{Bateman} == \Big\{\alpha == \sum_{j=1}^{m} A_j + \sum_{j=1}^{n} B_j - \sum_{j=1}^{p} C_j - \sum_{j=1}^{q} D_j,\ \beta == \sum_{j=1}^{m} A_j - \sum_{j=1}^{n} B_j - \sum_{j=1}^{p} C_j + \sum_{j=1}^{q} D_j,$$
$$\lambda == \mathrm{Re}\left[\sum_{j=1}^{m} a_j - \frac{m}{2} + \sum_{j=1}^{n} b_j - \frac{n}{2} - \sum_{j=1}^{p} c_j + \frac{p}{2} - \sum_{j=1}^{q} d_j + \frac{q}{2}\right],$$
$$\rho == \prod_{j=1}^{m} A_j^{A_j} \prod_{j=1}^{n} B_j^{-B_j} \prod_{j=1}^{p} C_j^{-C_j} \prod_{j=1}^{q} D_j^{D_j},\ \frac{\prod_{j=1}^{n} \mathrm{Gamma}[b_j + B_j s]\ \prod_{j=1}^{m} \mathrm{Gamma}[a_j - A_j s]}{\prod_{j=1}^{q} \mathrm{Gamma}[d_j + D_j s]\ \prod_{j=1}^{p} \mathrm{Gamma}[c_j - C_j s]}\ z^{-s}\Big\}$$

$$H[z] == H_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix}(a_j, \alpha_j)_{1,p} \\ (b_j, \beta_j)_{1,q}\end{matrix}\right] ==$$
$$\frac{1}{2\pi i}\int_{\mathcal{L}==C} \frac{\prod_{j=1}^{n} \mathrm{Gamma}[1 - a_j + \alpha_j s]\ \prod_{j=1}^{m} \mathrm{Gamma}[b_j - \beta_j s]}{\prod_{j=n+1}^{p} \mathrm{Gamma}[a_j - \alpha_j s]\ \prod_{j=m+1}^{q} \mathrm{Gamma}[1 - b_j + \beta_j s]}\ z^{s}\, ds ==$$
$$\frac{1}{2\pi i}\int_{\mathcal{L}==C} \frac{\prod_{j=1}^{m} \mathrm{Gamma}[b_j + \beta_j s]\ \prod_{j=1}^{n} \mathrm{Gamma}[1 - a_j - \alpha_j s]}{\prod_{j=n+1}^{p} \mathrm{Gamma}[a_j + \alpha_j s]\ \prod_{j=m+1}^{q} \mathrm{Gamma}[1 - b_j - \beta_j s]}$$
$$z^{-s}\, ds \quad (*\text{B.L.J.BRAAKSMA 1964}*)$$

$$\mu == (\delta) == \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j$$

$$\beta == (D) == \prod_{j=1}^{p} \alpha_j^{\alpha_j} \prod_{j=1}^{q} \beta_j^{-\beta_j}$$

$$\alpha == \sum_{j=1}^{p} a_j - \sum_{j=1}^{q} b_j + \frac{q - p + 1}{2}$$

FoxHCharacteristics

For descriptions of main terms asymptotics of FoxH function we will use several FoxHCharacteristics

$$\text{FoxHCharacteristics}[\{\text{Table}[\{a_i, \alpha_i\}, \{i, 1, n\}], \text{Table}[\{a_i, \alpha_i\}, \{i, n+1, p\}]\}, \{\text{Table}[\{b_i, \beta_i\}, \{i, 1, m\}], \text{Table}[\{b_i, \beta_i\}, \{i, m+1, q\}]\}, z] ==$$

$$\Big\{\frac{\left(\prod_{j=1}^{m} \text{Gamma}[b_j + \beta_j s]\right) \prod_{j=1}^{n} \text{Gamma}[1 - a_j - \alpha_j s]}{\left(\prod_{j=n+1}^{p} \text{Gamma}[a_j + \alpha_j s]\right) \prod_{j=m+1}^{q} \text{Gamma}[1 - b_j - \beta_j s]} z^{-s},$$

$$\alpha == \sum_{j=1}^{n} \alpha_j - \sum_{j=n+1}^{p} \alpha_j + \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q} \beta_j, \Delta == -\sum_{j=1}^{p} \alpha_j + \sum_{j=1}^{q} \beta_j,$$

$$\mu == \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j + \frac{p-q}{2}, \delta == \prod_{j=1}^{p} \alpha_j^{-\alpha_j} \prod_{j=1}^{q} \beta_j^{\beta_j}, H_{p,q}^{m,n}\left[z \,\middle|\, \begin{array}{c} (a_j, \alpha_j)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array}\right]\Big\}$$

$$a^* == \alpha H == \sum_{j=1}^{n} \alpha_j - \sum_{j=n+1}^{p} \alpha_j + \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q} \beta_j \quad (*7*) \; (*{==}\alpha \text{ in Bateman 49}*) \; (*{==}\alpha \text{ in Braaksma}*)$$

$$\delta == \beta H == \prod_{j=1}^{p} \alpha_j^{-\alpha_j} \prod_{j=1}^{q} \beta_j^{\beta_j} \quad (*9*) \; (*\rho{==}\tfrac{1}{\delta} \text{ in Bateman 49}*) \; (*{==}\tfrac{1}{\beta} \text{ in Braaksma}*)$$

$$\Delta == \mu H == \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j \quad (*8*) \; (*{==}-\beta \text{ in Bateman 49}*) \; (*{==}-\mu \text{ in Braaksma}*)$$

$$\mu == \delta H == \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j + \frac{p-q}{2} \quad (*10*)$$

$$(*\lambda{==}\text{Re}[\mu] \text{ in Bateman 49}*) \; (*{==}\tfrac{1}{2}-\alpha \text{ in Braaksma}*)$$

In[∘]:=

```
goodFoxHSubListQ[aA_] := MatrixQ[aA] && Union[Length /@ aA] ===
    {2}
```

In[∘]:=

```
FoxHCharacteristics[
  {ta2 : HoldPattern[Table[{a_, α_}, {i_, 1, m_}]] | aA2_?goodFoxHSubListQ| {},
   tb2 : HoldPattern[Table[{b_, β_}, {j_, 1, n_}]] |
     bB2_?goodFoxHSubListQ| {}},
  {tc2 : HoldPattern[Table[{c_, γ_}, {k_, 1, p_}]] | cC2_?goodFoxHSubListQ| {},
   td2 : HoldPattern[Table[{d_, δ_}, {l_, 1, q_}]] |
     dD2_?goodFoxHSubListQ| {}}, s_, z_] :=
```

$$\left\{\frac{\prod_{j=1}^{n}\text{Gamma}[b_j+\beta_j s]\ \prod_{j=1}^{m}\text{Gamma}[a_j-\alpha_j s]}{\prod_{j=1}^{q}\text{Gamma}[d_j+\delta_j s]\ \prod_{j=1}^{p}\text{Gamma}[c_j-\gamma_j s]}\, z^{-s},\right.$$

$$\alpha\alpha == \sum_{i=1}^{m}\alpha_i\ \text{Sum}[\alpha,\{i,1,m\}] + \text{Sum}[\beta,\{j,1,n\}] - \sum_{k=1}^{p}\gamma_k - \sum_{l=1}^{q}\delta_l,$$

$$\Delta\Delta == \sum_{i=1}^{m}\alpha_i - \sum_{j=1}^{n}\beta_j - \sum_{k=1}^{p}\gamma_k + \sum_{l=1}^{q}\delta_l,$$

$$\mu\mu == \sum_{i=1}^{m}a_i - \frac{m}{2} + \sum_{j=1}^{n}b_j - \frac{n}{2} - \sum_{k=1}^{p}c_k + \frac{p}{2} - \sum_{l=1}^{q}d_l + \frac{q}{2},$$

$$\left.\delta\delta == \left(\prod_{i=1}^{m}\alpha_i^{\alpha_i}\right)\left(\prod_{j=1}^{n}\beta_j^{-\beta_j}\right)\left(\prod_{k=1}^{p}\gamma_k^{-\gamma_k}\right)\prod_{l=1}^{q}\delta_l^{\delta_l}\right\} /.\ uu\_{}_{kk\_\ ll\_} :\to uu_{ll}$$

In[∘]:=

```
FoxHCharacteristics[{Table[{a_i, α_i}, {i, 1, m}], Table[{b_jj, β_jj}, {jj, 1, n}]},
  {Table[{c_k, γ_k}, {k, 1, p}], Table[{d_l, δ_l}, {l, 1, q}]}, s, z]
```

Out[∘]=

$$\left\{\frac{z^{-s}\prod_{jj=1}^{n}\text{Gamma}[b_{jj}+s\,\beta_{jj}]\ \prod_{jj=1}^{m}\text{Gamma}[a_{jj}-s\,\alpha_{jj}]}{\prod_{jj=1}^{q}\text{Gamma}[d_{jj}+s\,\delta_{jj}]\ \prod_{jj=1}^{p}\text{Gamma}[c_{jj}-s\,\gamma_{jj}]},\right.$$

$$\alpha\alpha == \sum_{i=1}^{m}\alpha_i + \sum_{jj=1}^{n}\beta_{jj} - \sum_{k=1}^{p}\gamma_k - \sum_{l=1}^{q}\delta_l,\ \Delta\Delta == \sum_{i=1}^{m}\alpha_i - \sum_{jj=1}^{n}\beta_{jj} - \sum_{k=1}^{p}\gamma_k + \sum_{l=1}^{q}\delta_l,$$

$$\left.\mu\mu == -\frac{m}{2} - \frac{n}{2} + \frac{p}{2} + \frac{q}{2} + \sum_{i=1}^{m}a_i + \sum_{jj=1}^{n}b_{jj} - \sum_{k=1}^{p}c_k - \sum_{l=1}^{q}d_l,\ \delta\delta == \left(\prod_{i=1}^{m}\alpha_i^{\alpha_i}\right)\left(\prod_{jj=1}^{n}\beta_{jj}^{-\beta_{jj}}\right)\left(\prod_{k=1}^{p}\gamma_k^{-\gamma_k}\right)\prod_{l=1}^{q}\delta_l^{\delta_l}\right\}$$

```
Inactive[FoxH][{Table[{a_i, α_i}, {i, 1, n}], Table[{a_i, α_i}, {i, n + 1, p}]},
  {Table[{b_i, β_i}, {i, 1, m}], Table[{b_i, β_i}, {i, m + 1, q}]}, z]
```

In[∘]:=

```
FoxHCharacteristics[{Table[{a_i, α_i}, {i, 1, n}], Table[{b_ii, β_ii}, {ii, 1, p}]},
  {Table[{c_k, γ_k}, {k, 1, p}], Table[{d_l, δ_l}, {l, 1, q}]}, s, z]
```

Out[∘]=

$$\left\{\frac{z^{-s}\prod_{ii=1}^{p}\text{Gamma}[b_{ii}+s\,\beta_{ii}]\prod_{ii=1}^{n}\text{Gamma}[a_{ii}-s\,\alpha_{ii}]}{\prod_{ii=1}^{q}\text{Gamma}[d_{ii}+s\,\delta_{ii}]\prod_{ii=1}^{p}\text{Gamma}[c_{ii}-s\,\gamma_{ii}]},\right.$$

$$\alpha\alpha == \sum_{ii=1}^{p}\beta_{ii}-\sum_{k=1}^{p}\gamma_k-\sum_{l=1}^{q}\delta_l+\sum_{i=1}^{n}\alpha_i\sum_{i=1}^{n}\alpha_i,\ \Delta\Delta == \sum_{i=1}^{n}\alpha_i-\sum_{ii=1}^{p}\beta_{ii}-\sum_{k=1}^{p}\gamma_k+\sum_{l=1}^{q}\delta_l,$$

$$\left.\mu\mu == -\frac{n}{2}+\frac{q}{2}+\sum_{i=1}^{n}a_i+\sum_{ii=1}^{p}b_{ii}-\sum_{k=1}^{p}c_k-\sum_{l=1}^{q}d_l,\ \delta\delta == \left(\prod_{i=1}^{n}\alpha_i^{\alpha_i}\right)\left(\prod_{ii=1}^{p}\beta_{ii}^{-\beta_{ii}}\right)\left(\prod_{k=1}^{p}\gamma_k^{-\gamma_k}\right)\prod_{l=1}^{q}\delta_l^{\delta_l}\right\}$$

Singular points of FoxH and MeijerG and O-terms near them

```
With[{m = 2, n = 3, p = 5, q = 5},
  FunctionSingularities[MeijerG[{Table[a_i, {i, 1, n}], Table[a_j, {j, 1 + n, p}]},
     {Table[b_k, {k, 1, m}], Table[b_l, {l, 1 + m, q}]}, z], z]]
```

••• FunctionSingularities: Warning: The set of singularities may be incomplete due to missing domain and singularity information for some of the functions involved.

Out[∘]=

```
z == 0 || (Im[z] == 0 && Re[z] ≤ 0) || Im[z]^2 + Re[z]^2 == 1
```

In[∘]:=

```
With[{m = 2, n = 3, p = 6, q = 5},
  FunctionSingularities[MeijerG[{Table[a_i, {i, 1, n}], Table[a_j, {j, 1 + n, p}]},
     {Table[b_k, {k, 1, m}], Table[b_l, {l, 1 + m, q}]}, z], z]]
```

••• FunctionSingularities: Warning: The set of singularities may be incomplete due to missing domain and singularity information for some of the functions involved.

Out[∘]=

```
z == 0 || (Im[z] == 0 && Re[z] ≤ 0)
```

In[∘]:=

```
FunctionSingularities[e^z, z, Complexes]
```

Out[∘]=

```
False
```

In[∘]:=

```
FunctionPoles[Gamma[z] Gamma[-z] Gamma[-z + 2], z]
```

Out[∘]=

$$\{\{1-2\,c_1 \text{ if } c_1\in\mathbb{Z}\,\&\&\,c_1\le -1, 2\}, \{2-2\,c_1 \text{ if } c_1\in\mathbb{Z}\,\&\&\,c_1\le 0, 2\}, \{2\,c_1 \text{ if } c_1\in\mathbb{Z}, \text{Indeterminate}\}, \{1+2\,c_1 \text{ if } c_1\in\mathbb{Z}\,\&\&\,c_1\ge 0, 1\}, \{1+2\,c_1 \text{ if } c_1\in\mathbb{Z}\,\&\&\,c_1\le -1, 1\}\}$$

FunctionAnalytic, FunctionPoles, FunctionDomain, FunctionPeriod, FunctionRange, FunctionSign, FunctionSingularities

FunctionSingularitiesOleg[MeijerG[{Table[a_i, {i, 1, n}], Table[a_j, {j, 1 + n, p}]}, {Table[b_k, {k, 1, m}], Table[b_l, {l, 1 + m, q}]}, z], z] ==

$$\begin{cases} 0 \wedge \infty & p<q \\ 0 \wedge (\text{Im}[z]==0\,\&\&\,\text{Re}[z]\le 0) \wedge \infty & p==q \\ \infty \wedge 0 & p>q \end{cases}$$

MeijerG[{Table[a_i, {i, 1, n}], Table[a_j, {j, 1 + n, p}]}, {Table[b_k, {k, 1, m}], Table[b_l, {l, 1 + m, q}]}, w z^g, r] ↔

$$\begin{cases} \sum_{k=1}^{m} z^{\frac{g\,b_k}{r}} & p\le q \\ e^{(-1)^{q-m-n} w^{-\frac{1}{r}} z^{-\frac{g}{r}}} z^{\frac{g\chi}{r}} + \sum_{k=1}^{m} z^{\frac{g\,b_k}{r}} & p==1+q \\ z^{\frac{g\chi}{r}} \text{Cos}\left[2\sqrt{(-1)^{q-m-n-1} w^{-\frac{1}{r}} z^{-\frac{g}{r}}}\right] + \sum_{k=1}^{m} z^{\frac{g\,b_k}{r}} & p==2+q \\ e^{(p-q)\left(-w^{\frac{1}{r}} z^{g/r}\right)^{\frac{1}{-p+q}}} z^{\frac{g\chi}{r}} + \sum_{k=1}^{m} z^{\frac{g\,b_k}{r}} & p\ge 3+q \end{cases} /;$$

$$\frac{g}{r}>0 \wedge \chi == \frac{1}{q-p}\left(\sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j + \frac{1+p-q}{2}\right) \wedge (\text{Abs}[z]\to 0)$$

$$\text{MeijerG}\left[\left\{\text{Table}\left[a_i,\{i,1,n\}\right],\text{Table}\left[a_j,\{j,1+n,p\}\right]\right\},\left\{\text{Table}\left[b_k,\{k,1,m\}\right],\text{Table}\left[b_l,\{l,1+m,q\}\right]\right\},w\,z^g,r\right]\leftrightarrow$$

$$\begin{cases} \sum_{k=1}^{m} z^{\frac{g\,b_k}{r}} & p\le q \\ e^{(-1)^{q-m-n}\,w^{-\frac{1}{r}}\,z^{-\frac{g}{r}}}\,z^{\frac{g\,\chi}{r}}+\sum_{k=1}^{m} z^{\frac{g\,b_k}{r}} & p==1+q \\ z^{\frac{g\,\chi}{r}}\,\text{Cos}\left[2\sqrt{(-1)^{q-m-n-1}\,w^{-\frac{1}{r}}\,z^{-\frac{g}{r}}}\right]+\sum_{k=1}^{m} z^{\frac{g\,b_k}{r}} & p==2+q \\ e^{(p-q)\left(-w^{\frac{1}{r}}\,z^{g/r}\right)^{\frac{1}{-p+q}}}\,z^{\frac{g\,\chi}{r}}+\sum_{k=1}^{m} z^{\frac{g\,b_k}{r}} & p\ge 3+q \end{cases}\ /;$$

$$\frac{g}{r}>0\land\chi==\frac{1}{q-p}\left(\sum_{j=1}^{q}b_j-\sum_{j=1}^{p}a_j+\frac{1+p-q}{2}\right)\land\left(\text{Abs}[z]\to 0\right)$$

$$\text{MeijerG}\left[\left\{\{a_1,\ldots,a_n\},\{a_{n+1},\ldots,a_q\}\right\},\left\{\{b_1,\ldots,b_m\},\{b_{m+1},\ldots,b_q\}\right\},z\right]\leftrightarrow$$

$$\begin{cases} 1+\left(1-(-1)^{p-m-n}\,z\right)^{\psi_p} & q==p\land\psi_p\ne 0 \\ 1+\text{Log}\left[1-(-1)^{p-m-n}\,z\right] & q==p\land\psi_p==0 \\ 1 & q\ne p \end{cases}\ /;$$

$$\psi_p==\sum_{j=1}^{p}\left(a_j-b_j\right)-1\land\left(z\to(-1)^{m+n-p}\right)$$

$$\text{MeijerG}\left[\left\{\text{Table}\left[a_i,\{i,1,n\}\right],\text{Table}\left[a_j,\{j,1+n,p\}\right]\right\},\left\{\text{Table}\left[b_k,\{k,1,m\}\right],\text{Table}\left[b_l,\{l,1+m,q\}\right]\right\},w\,z^g,r\right]\leftrightarrow$$

$$\begin{cases} \sum_{k=1}^{n} z^{\frac{g\,(a_k-1)}{r}} & q\le p \\ e^{(-1)^{p-m-n}\,w^{\frac{1}{r}}\,z^{\frac{q}{r}}}\,z^{\frac{g\,\chi}{r}}+\sum_{k=1}^{n} z^{\frac{g\,(a_k-1)}{r}} & q==1+p \\ z^{\frac{g\,\chi}{r}}\,\text{Cos}\left[2\sqrt{(-1)^{p-m-n-1}\,w^{\frac{1}{r}}\,z^{\frac{q}{r}}}\right]+\sum_{k=1}^{n} z^{\frac{g\,(a_k-1)}{r}} & q==2+p \\ e^{(q-p)\left(-w^{\frac{1}{r}}\,z^{\frac{q}{r}}\right)^{\frac{1}{q-p}}}\,z^{\frac{g\,\chi}{r}}+\sum_{k=1}^{n} z^{\frac{g\,(-1+a_k)}{r}} & q\ge 3+p \end{cases}\ /;$$

$$\frac{g}{r}>0\land\chi==\frac{1}{q-p}\left(\sum_{j=1}^{q}b_j-\sum_{j=1}^{p}a_j+\frac{1+p-q}{2}\right)\land\left(\text{Abs}[z]\to\infty\right)$$

Series expansion of FoxH[z] at 0

$$\text{Residue}\left[\frac{\left(\prod_{j=1}^{m}\text{Gamma}[b_j+\beta_j s]\right)\prod_{j=1}^{n}\text{Gamma}[1-a_j-\alpha_j s]}{\left(\prod_{j=n+1}^{p}\text{Gamma}[a_j+\alpha_j s]\right)\prod_{j=m+1}^{q}\text{Gamma}[1-b_j-\beta_j s]}z^{-s},\{s,b_{j,l}\}\right]==$$

$$\frac{(-1)^l}{l!\,\beta_j}\frac{\left(\prod_{i=1}^{m}\text{If}\left[i==j,1,\text{Gamma}\left[b_i+\beta_i\frac{-b_j-l}{\beta_j}\right]\right]\right)\prod_{i=1}^{n}\text{Gamma}\left[1-a_i-\alpha_i\frac{-b_j-l}{\beta_j}\right]}{\left(\prod_{i=n+1}^{p}\text{Gamma}\left[a_i+\alpha_i\frac{-b_j-l}{\beta_j}\right]\right)\prod_{i=m+1}^{q}\text{Gamma}\left[1-b_i-\beta_i\frac{-b_j-l}{\beta_j}\right]}$$

$$z^{-\frac{-b_j-l}{\beta_j}}\ /;\ b_{j,l}==\frac{-b_j-l}{\beta_j}$$

$$\beta_h(b_j+\lambda)\neq\beta_j(b_h+\nu)\ /;\ 1\le h\le m\,\&\&\,1\le j\le m\,\&\&\,\lambda,\nu==0,1,2,\ldots$$

Series expansion of FoxH[z] at ∞

$$H_{p,q}^{m,n}\left[x\,\middle|\,\begin{array}{c}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{array}\right]==$$

$$\text{Residue}\left[\frac{\left(\prod_{j=1}^{m}\text{Gamma}[b_j+\beta_j s]\right)\prod_{j=1}^{n}\text{Gamma}[1-a_j-\alpha_j s]}{\left(\prod_{j=n+1}^{p}\text{Gamma}[a_j+\alpha_j s]\right)\prod_{j=m+1}^{q}\text{Gamma}[1-b_j-\beta_j s]}z^{-s},\{s,a_{i,k}\}\right]==$$

$$\frac{(-1)^k}{k!\,\alpha_i}\frac{\left(\prod_{j=1}^{m}\text{Gamma}\left[b_j+\beta_j\frac{1-a_i+k}{\alpha_i}\right]\right)\prod_{j=1}^{n}\text{If}\left[j==i,1,\text{Gamma}\left[1-a_j-\alpha_j\frac{1-a_i+k}{\alpha_i}\right]\right]}{\left(\prod_{j=n+1}^{p}\text{Gamma}\left[a_j+\alpha_j\frac{1-a_i+k}{\alpha_i}\right]\right)\prod_{j=m+1}^{q}\text{Gamma}\left[1-b_j-\beta_j\frac{1-a_i+k}{\alpha_i}\right]}$$

$$z^{-\frac{1-a_i+k}{\alpha_i}}\ /;\ a_{i,k}==\frac{1-a_i+k}{\alpha_i}$$

$$\alpha_h(1-a_j+\lambda)\neq\alpha_j(1-a_h+\nu)\ /;\ 1\le h\le n\,\&\&\,1\le j\le n\,\&\&\,\lambda,\nu==0,1,2,\ldots$$

$$\text{MeijerG}[\{\{a_1,\ldots,a_n\},\{a_{n+1},\ldots,a_p\}\},\{\{b_1,\ldots,b_m\},\{b_{m+1},\ldots,b_q\}\},z,r]==$$

$$\pi^{m-1}\sum_{k=1}^{m}\frac{\prod_{j=1}^{n}\text{Gamma}[1+b_k-a_j]}{\prod_{j=1}^{m}\text{If}[j==k,1,\text{Sin}[\pi(b_j-b_k)]]\prod_{j=n+1}^{p}\text{Gamma}[a_j-b_k]}$$

$$z^{\frac{b_k}{r}}\,\text{HypergeometricPFQRegularized}\left[\{1+b_k-a_1,\ldots,1+b_k-a_p\},\right.$$

$$\left.\{1+b_k-b_1,\ldots,1+b_k-b_{k-1},1+b_k-b_{k+1},\ldots,1+b_k-b_q\},(-1)^{p-m-n}z^{\frac{1}{r}}\right]\ /;$$

$$(p<q\lor(p==q\land\text{Abs}[z]<1))\land$$

$$\forall_{\{j,k\},\{j,k\}\in\text{Integers}\land j\neq k\land 1\le j\le m\land 1\le k\le m}(!(b_j-b_k\in\text{Integers}))$$

List of 166+2 named functions as cases of FoxH and MeijerG functions

MittagLeffler2 cases

In[∘]:=

$$\text{MittagLefflerE}[a, b, z] == \sum_{k=0}^{\infty} \frac{z^k}{\text{Gamma}[a\,k + b]}$$

Out[∘]=

True

In[∘]:=

```
{#, ResourceFunction["FoxHForm"] [#, z] } & /@
  {MittagLefflerE[a, z], MittagLefflerE[a, b, z] } // TableForm
```

Out[∘]//TableForm=

MittagLefflerE[a, z]	FoxH[{ { {0, 1} }, { } }, { { {0, 1} }, { {0, a} } }, −z]
MittagLefflerE[a, b, z]	FoxH[{ { {0, 1} }, { } }, { { {0, 1} }, { {1 − b, a} } }, −z]

In[∘]:=

```
{#, ResourceFunction["MeijerGForm"] [#, z] } & /@
  {MittagLefflerE[a, z], MittagLefflerE[a, b, z] } // TableForm
```

Out[∘]//TableForm=

MittagLefflerE[a, z]	$\frac{(2\pi)^{\frac{1}{2}(-1+a)}\ \text{MeijerG}\left[\{\{0\},\{\}\},\left\{\{0\},\text{Table}\left[1-j-\frac{1+i}{a},\{i,0,-1+a\}\right]\right\},-a^{-a} z\right]}{\sqrt{a}}$ if $a \in \mathbb{Z}$ && $a > 0$
MittagLefflerE[a, b, z]	[▪] MeijerGForm(v1.2.0) + [MittagLefflerE[a, b, z], z]

FoxMeijer166

In[∘]:=

$$\text{FoxMeijer166} = \left\{a^z, e^z, \frac{1}{1-z}, \sqrt{z}, z^b, (1+z)^a, \text{Abs}[1-z]^a, \text{AiryAi}[z], \text{AiryAiPrime}[z],\right.$$

AiryBi[z], AiryBiPrime[z], AngerJ[a, z], AngerJ[a, b, z], ArcCos[z], ArcCosh[z],
ArcCot[z], ArcCoth[z], ArcCsc[z], ArcCsch[z], ArcSec[z], ArcSech[z], ArcSin[z],
ArcSinh[z], ArcTan[z], ArcTan[a, z], ArcTan[z, a], ArcTanh[z], BesselI[a, z],
BesselJ[a, z], BesselK[a, z], BesselY[a, z], Beta[z, a, b], Beta[c, z, a, b],
Beta[z, c, a, b], BetaRegularized[z, a, b], BetaRegularized[c, z, a, b],
BetaRegularized[z, c, a, b], BilateralHypergeometricPFQ[$\{a_1\}$, $\{b_1\}$, z],
BilateralHypergeometricPFQ[$\{a_1, a_2\}$, $\{b_1, b_2\}$, z],
BilateralHypergeometricPFQ[$\{a_1, a_2\}$, Table[b_i, {i, 1, q}], z],
BilateralHypergeometricPFQ[$\{a_1, a_2, a_3\}$, $\{b_1, b_2, b_3\}$, z],
BilateralHypergeometricPFQ[Table[a_i, {i, 1, p}], {b1, b2, b3}, z],

BilateralHypergeometricPFQ[Table[a_i, {i, 1, p}], Table[b_i, {i, 1, q}], z],
CarlsonRC[x, z], CarlsonRC[z, y], CarlsonRE[y, z], CarlsonRK[y, z],
ChebyshevT[a, z], ChebyshevU[a, z], Cos[z], Cosh[z], CoshIntegral[z],
CosIntegral[z], DawsonF[z], EllipticE[z], EllipticK[z], Erf[z], Erf[a, z],
Erf[z, b], Erfc[z], Erfi[z], ExpIntegralE[a, z], ExpIntegralEi[z],
Fibonacci[z], Fibonacci[a, z], FresnelC[z], FresnelF[z], FresnelG[z],
FresnelS[z], Gamma[a, z], Gamma[a, b, z], Gamma[a, z, b],
GammaRegularized[a, z], GammaRegularized[a, b, z], GammaRegularized[a, z, b],
GegenbauerC[a, b, z], HankelH1[a, z], HankelH2[a, z], Haversine[z],
$(1 - z)^a$ HeavisideTheta[1 - Abs[z]], $(-1 + z)^a$ HeavisideTheta[-1 + Abs[z]],
HermiteH[a, z], Hypergeometric0F1[a, z], Hypergeometric0F1Regularized[a, z],
Hypergeometric1F1[a, b, z], Hypergeometric1F1Regularized[a, b, z],
Hypergeometric2F1[a, b, c, z], Hypergeometric2F1Regularized[a, b, c, z],
HypergeometricPFQ[{a_1, a_2}, Table[b_j, {j, 1, q}], z],
HypergeometricPFQ[Table[a_j, {j, 1, p}], {b1, b2, b3}, z],
HypergeometricPFQ[Table[a_j, {j, 1, p}], Table[b_j, {j, 1, q}], z],
HypergeometricPFQRegularized[{a_1, a_2}, Table[b_j, {j, 1, q}], z],
HypergeometricPFQRegularized[Table[a_j, {j, 1, p}], {b1, b2, b3}, z],
HypergeometricPFQRegularized[Table[a_j, {j, 1, p}], Table[b_j, {j, 1, q}], z],
HypergeometricU[a, b, z], InverseHaversine[z], KelvinBei[0, z],
KelvinBei[a, z], KelvinBer[0, z], KelvinBer[a, z], KelvinKei[0, z],
KelvinKei[a, z], KelvinKer[0, z], KelvinKer[a, z], LaguerreL[a, z],
LaguerreL[a, b, z], LegendreP[ν, z], LegendreP[a, b, z], LegendreP[a, b, 2, z],
LegendreP[a, b, 3, z], LegendreQ[ν, z], LegendreQ[a, b, z], LegendreQ[a, b, 2, z],
LegendreQ[a, b, 3, z], Log[z], LucasL[z], LucasL[a, z], ParabolicCylinderD[a, z],

$$\begin{cases} 1 & \text{Abs}[z] > 1 \\ 0 & \text{True} \end{cases}, \begin{cases} 1 & \text{Abs}[z] < 1 \\ 0 & \text{True} \end{cases}, \begin{cases} (1-z)^a & \text{Abs}[z] < 1 \\ 0 & \text{True} \end{cases},$$

$$\text{AppellF3}\left[a, a1, b, b1, c, 1 - \frac{1}{z}, 1 - z\right]\left(\begin{cases} (1-z)^{-1+c} & \text{Abs}[z] < 1 \\ 0 & \text{True} \end{cases}\right),$$

$$\text{AppellF3}\left[a, a1, b, b1, c, 1 - z, 1 - \frac{1}{z}\right]\left(\begin{cases} (1-z)^{-1+c} & \text{Abs}[z] < 1 \\ 0 & \text{True} \end{cases}\right),$$

$$\text{Hypergeometric2F1}\left[a, b, c, 1 - \frac{1}{z}\right]\left(\begin{cases} (1-z)^{-1+c} & \text{Abs}[z] < 1 \\ 0 & \text{True} \end{cases}\right),$$

$$\text{Hypergeometric2F1}[a, b, c, 1 - z]\left(\begin{cases} (1-z)^{-1+c} & \text{Abs}[z] < 1 \\ 0 & \text{True} \end{cases}\right),$$

$$\begin{cases} (-1+z)^{a} & \text{Abs}[z]>1 \\ 0 & \text{True} \end{cases},$$

$$\text{AppellF3}\left[a, a1, b, b1, c, 1-\frac{1}{z}, 1-z\right]\left(\begin{cases} (-1+z)^{-1+c} & \text{Abs}[z]>1 \\ 0 & \text{True} \end{cases}\right),$$

$$\text{AppellF3}\left[a, a1, b, b1, c, 1-z, 1-\frac{1}{z}\right]\left(\begin{cases} (-1+z)^{-1+c} & \text{Abs}[z]>1 \\ 0 & \text{True} \end{cases}\right),$$

$$\text{Hypergeometric2F1}\left[a, b, c, 1-\frac{1}{z}\right]\left(\begin{cases} (-1+z)^{-1+c} & \text{Abs}[z]>1 \\ 0 & \text{True} \end{cases}\right),$$

$$\text{Hypergeometric2F1}[a, b, c, 1-z]\left(\begin{cases} (-1+z)^{-1+c} & \text{Abs}[z]>1 \\ 0 & \text{True} \end{cases}\right), \text{PolyLog}[2, z],$$

PolyLog[a, z], ScorerGi[z], ScorerGiPrime[z], ScorerHi[z], ScorerHiPrime[z],

$\text{Abs}[1-z]^{a}\,\text{Sign}[1-z]$, $\text{Abs}[1-z]^{a}\,\text{Sign}[-1+z]$, Sin[z], Sinc[z], Sinh[z],

SinhIntegral[z], SinIntegral[z], SphericalBesselJ[a, z], SphericalBesselY[a, z],

SphericalHankelH1[a, z], SphericalHankelH2[a, z], StruveH[ν, z], StruveL[ν, z],

UnitStep[z], UnitStep[1 − Abs[z]], $(1-z)^{a}$ UnitStep[1 − Abs[z]],

$$(1-z)^{-1+c}\,\text{AppellF3}\left[a, a1, b, b1, c, 1-\frac{1}{z}, 1-z\right]\text{UnitStep}[1-\text{Abs}[z]],$$

$$(1-z)^{-1+c}\,\text{AppellF3}\left[a, a1, b, b1, c, 1-z, 1-\frac{1}{z}\right]\text{UnitStep}[1-\text{Abs}[z]],$$

UnitStep[−1 + Abs[z]], $(-1+z)^{a}$ UnitStep[−1 + Abs[z]],

$$(-1+z)^{-1+c}\,\text{AppellF3}\left[a, a1, b, b1, c, 1-\frac{1}{z}, 1-z\right]\text{UnitStep}[-1+\text{Abs}[z]],$$

$$(-1+z)^{-1+c}\,\text{AppellF3}\left[a, a1, b, b1, c, 1-z, 1-\frac{1}{z}\right]\text{UnitStep}[-1+\text{Abs}[z]],$$

WeberE[ν, z], WeberE[ν, a, z], WhittakerM[a, b, z],

$e^{-z/2}$ WhittakerM[a, b, z], $e^{z/2}$ WhittakerM[a, b, z], WhittakerW[a, b, z],

$e^{-z/2}$ WhittakerW[a, b, z], $e^{z/2}$ WhittakerW[a, b, z]};

In[∘]:=

```
FoxMeijer166 // TraditionalForm
```

Out[∘]//TraditionalForm=

$$\Big\{a^z,\ e^z,\ \frac{1}{1-z},\ \sqrt{z},\ z^b,\ (z+1)^a,\ |1-z|^a,\ \mathrm{Ai}(z),\ \mathrm{Ai}'(z),\ \mathrm{Bi}(z),\ \mathrm{Bi}'(z),\ \mathbf{J}_a(z),\ \mathbf{J}_a^b(z),\ \cos^{-1}(z),\ \cosh^{-1}(z),\ \cot^{-1}(z),$$

$$\coth^{-1}(z),\ \csc^{-1}(z),\ \mathrm{csch}^{-1}(z),\ \sec^{-1}(z),\ \mathrm{sech}^{-1}(z),\ \sin^{-1}(z),\ \sinh^{-1}(z),\ \tan^{-1}(z),\ \tan^{-1}(a,z),\ \tan^{-1}(z,a),$$

$$\tanh^{-1}(z),\ I_a(z),\ J_a(z),\ K_a(z),\ Y_a(z),\ \mathrm{B}_z(a,b),\ B_{(c,z)}(a,b),\ B_{(z,c)}(a,b),\ I_z(a,b),\ I_{(c,z)}(a,b),\ I_{(z,c)}(a,b),$$

$${}_1H_1(a_1;b_1;z),\ {}_2H_2(a_1,a_2;b_1,b_2;z),\ \mathrm{BilateralHypergeometricPFQ}[\{a_1,a_2\},\mathrm{Table}[b_i,\{i,1,q\}],z],$$

$${}_3H_3(a_1,a_2,a_3;b_1,b_2,b_3;z),\ \mathrm{BilateralHypergeometricPFQ}[\mathrm{Table}[a_i,\{i,1,p\}],\{\mathrm{b1},\mathrm{b2},\mathrm{b3}\},z],$$

$$\mathrm{BilateralHypergeometricPFQ}[\mathrm{Table}[a_i,\{i,1,p\}],\mathrm{Table}[b_i,\{i,1,q\}],z],\ R_C(x,z),\ R_C(z,y),\ R_E(y,z),$$

$$R_K(y,z),\ T_a(z),\ U_a(z),\ \cos(z),\ \cosh(z),\ \mathrm{Chi}(z),\ \mathrm{Ci}(z),\ F(z),\ E(z),\ K(z),\ \mathrm{erf}(z),\ \mathrm{erf}(a,z),\ \mathrm{erf}(z,b),\ \mathrm{erfc}(z),$$

$$\mathrm{erfi}(z),\ E_a(z),\ \mathrm{Ei}(z),\ F_z,\ F_a(z),\ C(z),\ F(z),\ G(z),\ S(z),\ \Gamma(a,z),\ \Gamma(a,b,z),\ \Gamma(a,z,b),\ Q(a,z),\ Q(a,b,z),$$

$$Q(a,z,b),\ C_a^{(b)}(z),\ H_a^{(1)}(z),\ H_a^{(2)}(z),\ \mathrm{hav}(z),\ (1-z)^a\,\theta(1-|z|),\ (z-1)^a\,\theta(|z|-1),\ H_a(z),\ {}_0F_1(;a;z),\ {}_0\tilde{F}_1(;a;z),$$

$${}_1F_1(a;b;z),\ {}_1\tilde{F}_1(a;b;z),\ {}_2F_1(a,b;c;z),\ {}_2\tilde{F}_1(a,b;c;z),\ \mathrm{HypergeometricPFQ}\big[\{a_1,a_2\},\mathrm{Table}\big[b_j,\{j,1,q\}\big],z\big],$$

$$\mathrm{HypergeometricPFQ}\big[\mathrm{Table}\big[a_j,\{j,1,p\}\big],\{\mathrm{b1},\mathrm{b2},\mathrm{b3}\},z\big],$$

$$\mathrm{HypergeometricPFQ}\big[\mathrm{Table}\big[a_j,\{j,1,p\}\big],\mathrm{Table}\big[b_j,\{j,1,q\}\big],z\big],$$

$$\mathrm{HypergeometricPFQRegularized}\big[\{a_1,a_2\},\mathrm{Table}\big[b_j,\{j,1,q\}\big],z\big],$$

$$\mathrm{HypergeometricPFQRegularized}\big[\mathrm{Table}\big[a_j,\{j,1,p\}\big],\{\mathrm{b1},\mathrm{b2},\mathrm{b3}\},z\big],$$

$$\mathrm{HypergeometricPFQRegularized}\big[\mathrm{Table}\big[a_j,\{j,1,p\}\big],\mathrm{Table}\big[b_j,\{j,1,q\}\big],z\big],\ U(a,b,z),\ \mathrm{hav}^{-1}(z),$$

$$\mathrm{bei}_0(z),\ \mathrm{bei}_a(z),\ \mathrm{ber}_0(z),\ \mathrm{ber}_a(z),\ \mathrm{kei}_0(z),\ \mathrm{kei}_a(z),\ \mathrm{ker}_0(z),\ \mathrm{ker}_a(z),\ L_a(z),\ L_a^b(z),\ P_\nu(z),\ P_a^b(z),\ P_a^b(z),\ P_a^b(z),$$

$$Q_\nu(z),\ Q_a^b(z),\ Q_a^b(z),\ Q_a^b(z),\ \log(z),\ L_z,\ L_a(z),\ D_a(z),\ \begin{cases}1 & |z|>1\\ 0 & \text{True}\end{cases},\ \begin{cases}1 & |z|<1\\ 0 & \text{True}\end{cases},\ \begin{cases}(1-z)^a & |z|<1\\ 0 & \text{True}\end{cases},$$

$$F_3\left(a,\mathrm{a1};b,\mathrm{b1};c;1-\frac{1}{z},1-z\right)\left(\begin{cases}(1-z)^{c-1} & |z|<1\\ 0 & \text{True}\end{cases}\right),\ F_3\left(a,\mathrm{a1};b,\mathrm{b1};c;1-z,1-\frac{1}{z}\right)\left(\begin{cases}(1-z)^{c-1} & |z|<1\\ 0 & \text{True}\end{cases}\right),$$

$${}_2F_1\left(a,b;c;1-\frac{1}{z}\right)\left(\begin{cases}(1-z)^{c-1} & |z|<1\\ 0 & \text{True}\end{cases}\right),\ {}_2F_1(a,b;c;1-z)\left(\begin{cases}(1-z)^{c-1} & |z|<1\\ 0 & \text{True}\end{cases}\right),\ \begin{cases}(z-1)^a & |z|>1\\ 0 & \text{True}\end{cases},$$

$$F_3\left(a,\mathrm{a1};b,\mathrm{b1};c;1-\frac{1}{z},1-z\right)\left(\begin{cases}(z-1)^{c-1} & |z|>1\\ 0 & \text{True}\end{cases}\right),\ F_3\left(a,\mathrm{a1};b,\mathrm{b1};c;1-z,1-\frac{1}{z}\right)\left(\begin{cases}(z-1)^{c-1} & |z|>1\\ 0 & \text{True}\end{cases}\right),$$

$${}_2F_1\left(a,b;c;1-\frac{1}{z}\right)\left(\begin{cases}(z-1)^{c-1} & |z|>1\\ 0 & \text{True}\end{cases}\right),\ {}_2F_1(a,b;c;1-z)\left(\begin{cases}(z-1)^{c-1} & |z|>1\\ 0 & \text{True}\end{cases}\right),\ \mathrm{Li}_2(z),\ \mathrm{Li}_a(z),\ \mathrm{Gi}(z),$$

$$\mathrm{Gi}'(z),\ \mathrm{Hi}(z),\ \mathrm{Hi}'(z),\ |1-z|^a\,\mathrm{sgn}(1-z),\ |1-z|^a\,\mathrm{sgn}(z-1),\ \sin(z),\ \mathrm{sinc}(z),\ \sinh(z),\ \mathrm{Shi}(z),\ \mathrm{Si}(z),\ j_a(z),\ y_a(z),$$

$$h_a^{(1)}(z),\ h_a^{(2)}(z),\ \mathbf{H}_\nu(z),\ \mathbf{L}_\nu(z),\ \theta(z),\ \theta(1-|z|),\ (1-z)^a\,\theta(1-|z|),\ (1-z)^{c-1}\,F_3\left(a,\mathrm{a1};b,\mathrm{b1};c;1-\frac{1}{z},1-z\right)\theta(1-|z|),$$

$$(1-z)^{c-1}\,F_3\left(a,\mathrm{a1};b,\mathrm{b1};c;1-z,1-\frac{1}{z}\right)\theta(1-|z|),\ \theta(|z|-1),\ (z-1)^a\,\theta(|z|-1),$$

$$(z-1)^{c-1}\,F_3\left(a,\mathrm{a1};b,\mathrm{b1};c;1-\frac{1}{z},1-z\right)\theta(|z|-1),\ (z-1)^{c-1}\,F_3\left(a,\mathrm{a1};b,\mathrm{b1};c;1-z,1-\frac{1}{z}\right)\theta(|z|-1),$$

$$\mathbf{E}_\nu(z),\ \mathbf{E}_\nu^a(z),\ M_{a,b}(z),\ e^{-z/2}\,M_{a,b}(z),\ e^{z/2}\,M_{a,b}(z),\ W_{a,b}(z),\ e^{-z/2}\,W_{a,b}(z),\ e^{z/2}\,W_{a,b}(z)\Big\}$$

Test of MeijerGForm[FoxMeijer166]

```
{#, ResourceFunction["MeijerGForm"][#, z]} & /@
    Table[FoxMeijer166[[k]], {k, 1, 166}] // EchoTiming // TableForm (*2.45 Sec*)
```

2.45394

Out[∘]//TableForm=

a^z	MeijerG[{{}, {}},
e^z	MeijerG[{{}, {}},
$\frac{1}{1-z}$	π MeijerG[{{0}, {$\frac{1}{2}$
$\sqrt{z}$	MeijerG[{{}, {}},
z^b	MeijerG[{{}, {}},
$(1+z)^a$	$\sum_{k=0}^{a} \frac{1}{k!}$ { Pochhammer[… MeijerG[… ; $\frac{\text{MeijerG[\{\{1+a\},\{\},\{}}{\text{Gamma[-a}}$
Abs[1 − z]^a	$\frac{\pi \text{ Sec}[\frac{a\pi}{2}] \text{ MeijerG}[\{\{1+a\}}{\text{Gamma}}$
AiryAi[z]	$\frac{\text{MeijerG}[\{\{\},\{\}\},\{\{0,\frac{1}{3}\},\{\}}{2\times 3^{1/6}\pi}$
AiryAiPrime[z]	$-\frac{3^{1/6} \text{ MeijerG}[\{\{\},\{\}\},\{\{}{2\pi}$
AiryBi[z]	$\frac{2\pi \text{ MeijerG}[\{\{\},\{\frac{1}{6},\frac{2}{3}\}\},\{\{0,}{3^{1/6}}$
AiryBiPrime[z]	$-2\times 3^{1/6}\pi$ MeijerG[
AngerJ[a, z]	MeijerG[{{0, $\frac{1}{2}$}, {
AngerJ[a, b, z]	2^b MeijerG[{{−$\frac{b}{2}$, $\frac{1-}{2}$
ArcCos[z]	$\frac{1}{2}\pi$ (MeijerG[{{},
ArcCosh[z]	$\frac{\pi\sqrt{-1+z}\ (\text{MeijerG}[\{\{\},\{\}}{}$
ArcCot[z]	$\frac{1}{2}$ i (MeijerG[{{1, 1
ArcCoth[z]	$\frac{1}{2}$ (−MeijerG[{{1, 1
ArcCsc[z]	$-\frac{i \text{ MeijerG}[\{\{1,1\},\{\}\},\{\{}{2\sqrt{\pi}}$
ArcCsch[z]	$\frac{\text{MeijerG}[\{\{1,1\},\{\}\},\{\{\frac{1}{2}\},\{}{2\sqrt{\pi}}$
ArcSec[z]	$\frac{1}{2}\pi$ (MeijerG[{{},

ArcSech[z]	$\frac{\pi \sqrt{-1+\frac{1}{z}}\ (\text{MeijerG}[\{\{\},\{\}}{}$
ArcSin[z]	$-\frac{i\ \text{MeijerG}[\{\{1,1\},\{\}\},\{\{}{2\sqrt{\pi}}$
ArcSinh[z]	$\frac{\text{MeijerG}[\{\{1,1\},\{\}\},\{\{\frac{1}{2}\},\{}{2\sqrt{\pi}}$
ArcTan[z]	$\frac{1}{2}\ \text{MeijerG}[\{\{\frac{1}{2},1\},\{$
ArcTan[a, z]	(ArcTan[a, z] + i L
ArcTan[z, a]	(ArcTan[z, a] + i L
ArcTanh[z]	$-\frac{1}{2}\ i\ \text{MeijerG}[\{\{\frac{1}{2},1$
BesselI[a, z]	$\pi\ \text{MeijerG}[\{\{\},\{\frac{1+a}{2}$
BesselJ[a, z]	MeijerG[{{}, {}},
BesselK[a, z]	$\frac{1}{2}\ \text{MeijerG}[\{\{\},\{\}\}$
BesselY[a, z]	$\text{MeijerG}[\{\{\},\{\frac{1}{2}\ (-$
Beta[z, a, b]	$\frac{(-z)^{-a}\ z^{a}\ \text{MeijerG}[\{\{1,a+b}{\text{Gamma}[1}$
Beta[c, z, a, b]	−Beta[c, a, b] (Me
Beta[z, c, a, b]	Beta[c, a, b] (Meij
BetaRegularized[z, a, b]	$\frac{(-z)^{-a}\ z^{a}\ \text{MeijerG}[\{\{1,a+b}{\text{Beta}[a,b]\ \text{Gam}}$
BetaRegularized[c, z, a, b]	$\frac{-\text{Beta}[c,a,b]\ (\text{MeijerG}[\{\{}{}$
BetaRegularized[z, c, a, b]	$\frac{\text{Beta}[c,a,b]\ (\text{MeijerG}[\{\{\},}{}$
BilateralHypergeometricPFQ[{a_1}, {b_1}, z]	[▪] MeijerGForm(v1.2.
BilateralHypergeometricPFQ[{a_1, a_2}, {b_1, b_2}, z]	[▪] MeijerGForm(v1.2.
BilateralHypergeometricPFQ[{a_1, a_2}, Table[b_i, {i, 1, q}], z]	[▪] MeijerGForm(v1.2.
BilateralHypergeometricPFQ[{a_1, a_2, a_3}, {b_1, b_2, b_3}, z]	[▪] MeijerGForm(v1.2.
BilateralHypergeometricPFQ[Table[a_i, {i, 1, p}], {b1, b2, b3}, z]	[▪] MeijerGForm(v1.2.
BilateralHypergeometricPFQ[Table[a_i, {i, 1, p}], Table[b_i, {i, 1, q}], z]	[▪] MeijerGForm(v1.2.
CarlsonRC[x, z]	$\frac{\text{MeijerG}[\{\{0,\frac{1}{2}\},\{\}\},\{\{0,0\}}{2\sqrt{\pi}\ \sqrt{x}}$
CarlsonRC[z, y]	$\frac{\text{MeijerG}[\{\{\frac{1}{2},\frac{1}{2}\},\{\}\},\{\{0,\frac{1}{2}\},}{2\sqrt{\pi}\ \sqrt{y}}$
CarlsonRE[y, z]	$-\frac{\sqrt{y}\ \text{MeijerG}[\{\{\frac{3}{2},\frac{1}{2}\},\{\}\},\{}{\pi^2}$
CarlsonRK[y, z]	$\frac{\text{MeijerG}[\{\{\frac{1}{2},\frac{1}{2}\},\{\}\},\{\{0,0\}}{\pi^2\ \sqrt{y}}$

ChebyshevT[a, z]	$\sum_{k=0}^{a} \frac{1}{k!}$ (−1)^{k} √… Hypergeom… (MeijerG[… MeijerG… − a Sin[a π] MeijerG[{…
ChebyshevU[a, z]	$\sum_{k=0}^{a} \frac{1}{2\,k!}$ (−1)^{k} (… Hypergeom… {$\frac{3}{2}$, 1 − k… MeijerG… − Sin[a π] MeijerG[{{1…
Cos[z]	$\sqrt{\pi}$ MeijerG[{{}, {…
Cosh[z]	$\sqrt{\pi}$ MeijerG[{{}, {…
CoshIntegral[z]	$-\frac{1}{2}\pi^{3/2}$ MeijerG[{{…
CosIntegral[z]	$-\frac{1}{2}\sqrt{\pi}$ MeijerG[{{…
DawsonF[z]	$\frac{1}{2}\sqrt{\pi}$ MeijerG[{{$\frac{1}{2}$}…
EllipticE[z]	$-\frac{1}{4}$ MeijerG[{{$\frac{1}{2}$, $\frac{3}{2}$}…
EllipticK[z]	$\frac{1}{2}$ MeijerG[{{$\frac{1}{2}$, $\frac{1}{2}$}, {…
Erf[z]	MeijerG[{{1}, {}}, {{$\frac{1}{2}$}, {0… / $\sqrt{\pi}$
Erf[a, z]	Erf[a] (MeijerG[{…
Erf[z, b]	−Erf[b] (MeijerG[…
Erfc[z]	MeijerG[{{}, {1}}, {{0, $\frac{1}{2}$}, {… / $\sqrt{\pi}$
Erfi[z]	− *i* MeijerG[{{1}, {}}, {{$\frac{1}{2}$}… / $\sqrt{\pi}$
ExpIntegralE[a, z]	MeijerG[{{}, {a}}…
ExpIntegralEi[z]	−MeijerG[{{}, {1…
Fibonacci[z]	2 MeijerG[{{}, {}}, {{0}, {…
Fibonacci[a, z]	$\sum_{k=0}^{\text{Floor}[\frac{1}{2}(-1+a)]}$ Bin… (MeijerG[{{… {{−2 k… Sin[a π] MeijerG[{{$\frac{1+a}{2}$…

FresnelC[z]	$\frac{e^{-\frac{i\pi}{4}}\pi\,\text{MeijerG}[\{\{\},\{1\}\},\{\ldots}{\sqrt{2}\ldots}$
FresnelF[z]	$\frac{\text{MeijerG}[\{\{\frac{3}{4}\},\{\}\},\{\{0,\frac{1}{2},\frac{3}{4}\},\ldots}{2\sqrt{2}\,\pi^{3/2}}$
FresnelG[z]	$\frac{\text{MeijerG}[\{\{\frac{1}{4}\},\{\}\},\{\{0,\frac{1}{4},\frac{1}{2}\},\ldots}{2\sqrt{2}\,\pi^{3/2}}$
FresnelS[z]	$\frac{e^{-\frac{3i\pi}{4}}\pi\,\text{MeijerG}[\{\{\},\{1\}\},\{\ldots}{\sqrt{2}\ldots}$
Gamma[a, z]	MeijerG[{{}, {1}}
Gamma[a, b, z]	– MeijerG[{{}, {1}
Gamma[a, z, b]	MeijerG[{{}, {1}}
GammaRegularized[a, z]	$\frac{\text{MeijerG}[\{\{\},\{1\}\},\{\{0,a\},\ldots}{\text{Gamma}[a]}$
GammaRegularized[a, b, z]	$-\frac{\text{MeijerG}[\{\{\},\{1\}\},\{\{0,a\ldots}{\text{Gamma}[a]}$
GammaRegularized[a, z, b]	$\frac{\text{MeijerG}[\{\{\},\{1\}\},\{\{0,a\},\ldots}{\text{Gamma}[a]}$
GegenbauerC[a, b, z]	$\sum_{k=0}^{a}\frac{1}{k!\,a!\,\text{Gamma}[b]\ldots}$ Hypergeom… $\{\frac{1}{2}+b, 1\ldots$ Meijer… ; $\sum_{k=0}^{\infty}\frac{1}{k!\,\text{Gamma}[1+a]\ldots}$ Hypergeom… $\{\frac{1}{2}+b, 1\ldots$ Meijer…
HankelH1[a, z]	MeijerG[{{}, {}},
HankelH2[a, z]	MeijerG[{{}, {}},
Haversine[z]	$\frac{1}{2}\sqrt{\pi}$ MeijerG[{{1}
(1 – z)^a HeavisideTheta[1 – Abs[z]]	[▪] MeijerGForm(v1.2.
(–1 + z)^a HeavisideTheta[–1 + Abs[z]]	[▪] MeijerGForm(v1.2.
HermiteH[a, z]	[▪] MeijerGForm(v1.2.
Hypergeometric0F1[a, z]	Gamma[a] MeijerG
Hypergeometric0F1Regularized[a, z]	MeijerG[{{}, {}},
Hypergeometric1F1[a, b, z]	$\frac{\pi\,\text{Gamma}[b]\,\text{MeijerG}[\{\{1-\ldots}{\text{Gamm}\ldots}$
Hypergeometric1F1Regularized[a, b, z]	$\frac{\pi\,\text{MeijerG}[\{\{1-a\},\{\frac{1}{2}\}\},\{\{0\ldots}{\text{Gamma}[a]}$
Hypergeometric2F1[a, b, c, z]	$\frac{\text{Gamma}[c]\,\text{MeijerG}[\{\{1-a,\ldots}{\text{Gamma}[a]}$
Hypergeometric2F1Regularized[a, b, c, z]	$\frac{\text{MeijerG}[\{\{1-a,1-b\},\{\}\},\{\ldots}{\text{Gamma}[a]\,\text{Gam}\ldots}$

HypergeometricPFQ[$\{a_1, a_2\}$, Table[b_j, {j, 1, q}], z]	MeijerG[{ {1−a$_1$,1−a$_2$}, { } … / Ga…
HypergeometricPFQ[Table[a_j, {j, 1, p}], {b1, b2, b3}, z]	… / Gamma[b1] Gamma[b2] G…
HypergeometricPFQ[Table[a_j, {j, 1, p}], Table[b_j, {j, 1, q}], z]	MeijerG[{Table[1−a$_j$,{j,1,p…
HypergeometricPFQRegularized[$\{a_1, a_2\}$, Table[b_j, {j, 1, q}], z]	MeijerG[{ {1−a$_1$,1−a$_2$}, { } … / Gamma[a$_1$…
HypergeometricPFQRegularized[Table[a_j, {j, 1, p}], {b1, b2, b3}, z]	MeijerG[{Table[1−a$_j$,{j,1,p… / $\prod_{j=}^{p}$…
HypergeometricPFQRegularized[Table[a_j, {j, 1, p}], Table[b_j, {j, 1, q}], z]	MeijerG[{Table[1−a$_j$,{j,1,p… / $\prod_{j=}^{p}$…
HypergeometricU[a, b, z]	MeijerG[{ {1−a}, { } }, { {0,1… / Gamma[a] Gamma[1…
InverseHaversine[z]	$-\frac{\sqrt{-z}\ \text{MeijerG}[\{\{1,1\},\{\}\ldots}{\sqrt{\pi}\ \sqrt{z}}$
KelvinBei[0, z]	π MeijerG[{ { }, { } …
KelvinBei[a, z]	π MeijerG[{ { }, {a} …
KelvinBer[0, z]	π MeijerG[{ { }, { } …
KelvinBer[a, z]	π MeijerG[{ { }, { $\frac{1}{2}$ (…
KelvinKei[0, z]	$-\frac{1}{4}$ MeijerG[{ { }, { …
KelvinKei[a, z]	$-\frac{1}{4}$ MeijerG[{ { }, { $\frac{a}{2}$ …
KelvinKer[0, z]	$\frac{1}{4}$ MeijerG[{ { }, { } …
KelvinKer[a, z]	$\frac{1}{4}$ MeijerG[{ { }, { $\frac{1+a}{2}$ …
LaguerreL[a, z]	$\sum_{k=0}^{a} \frac{1}{(k!)^2}$ … { Pochhammer[… MeijerG[… Gamma[1 + a] M…
LaguerreL[a, b, z]	$\sum_{k=0}^{a} \frac{1}{k!\ a!\ \text{Gamma}[1+\ldots}$ … { Pochhamme… MeijerG[… Gamma[1 + a + b…
LegendreP[v, z]	$\sum_{k=0}^{\text{Floor}[\frac{v}{2}]}$ (−1)^k 2… { $\sum_{k=0}^{\infty} \frac{1}{k!}$ (−1)^k Hy… (MeijerG[{ { … MeijerG[… ! ((v ∈ ℤ && v ≥ …
LegendreP[a, b, z]	$-\frac{(1-z)^{-b/2}\ (1+z)^{b/2}\ \text{Sin}[\ldots}{\ldots}$

LegendreP[a, b, 2, z]	$\Big\{$ $\sum_{k=0}^{a}$ (−1)^{k} Gamr … MeijerG[… $\sum_{j=0}^{k} \frac{1}{j!\,(-j+k)!}$ H … {1 − b, … $\sum_{k=0}^{\infty}$ (−1)^{k} Gamr … MeijerG[… $\sum_{j=0}^{k} \frac{1}{j!\,(-j+k)!}$ H … {1 − b,
LegendreP[a, b, 3, z]	$-\frac{(-1+z)^{-b/2}\,(1+z)^{b/2}\,\text{Sir}}{}$
LegendreQ[v, z]	$\frac{1}{2}$ (MeijerG[{{1 + v … Cos[π v] M
LegendreQ[a, b, z]	$-\frac{1}{2}$ Csc[b π] Sin[a … (− (1 − z)$^{b/2}$ (1 … { } }, … MeijerG[
LegendreQ[a, b, 2, z]	$-\frac{1}{2}$ Csc[b π] Sin[a … (− (1 − z)$^{b/2}$ (1 … { } }, … MeijerG[
LegendreQ[a, b, 3, z]	$-\frac{1}{2}$ $e^{i\, b\, \pi}$ Csc[b π] S … (− (−1 + z)$^{b/2}$ … { } }, … MeijerG[
Log[z]	−MeijerG[{{0, 0},
LucasL[z]	$\frac{1}{2}$ (2 MeijerG[{{ },

LucasL[a, z]	$\sum_{k=0}^{\text{Floor}[\frac{a}{2}]} \frac{1}{k!} 2^{1+2k-}$ … $\begin{cases} \text{Hypergeon…} \\ (\text{MeijerG}[… \\ \{\{- … \end{cases}$ $-\frac{a \operatorname{Sin}[a\pi] \text{ MeijerG}[\{…}{…}$
ParabolicCylinderD[a, z]	$(-1)^{\text{Floor}[\frac{a}{2}]} 2^{\text{Floor}[\frac{1}{…}]}$ … $\sum_{k=0}^{\text{Floor}[\frac{a}{2}]} \frac{2^k \text{ Pochha…}}{…}$ if $a \in \mathbb{Z}$ && $a \geq 0$
$\begin{cases} 1 & \text{Abs}[z] > 1 \\ 0 & \text{True} \end{cases}$	[▪] MeijerGForm(v1.2.…
$\begin{cases} 1 & \text{Abs}[z] < 1 \\ 0 & \text{True} \end{cases}$	[▪] MeijerGForm(v1.2.…
$\begin{cases} (1-z)^a & \text{Abs}[z] < 1 \\ 0 & \text{True} \end{cases}$	[▪] MeijerGForm(v1.2.…
$\text{AppellF3}[a, a1, b, b1, c, 1-\frac{1}{z}, 1-z] \left(\begin{cases} (1-z)^{-1+c} & \text{Abs}[z] < 1 \\ 0 & \text{True} \end{cases}\right)$	[▪] MeijerGForm(v1.2.…
$\text{AppellF3}[a, a1, b, b1, c, 1-z, 1-\frac{1}{z}] \left(\begin{cases} (1-z)^{-1+c} & \text{Abs}[z] < 1 \\ 0 & \text{True} \end{cases}\right)$	[▪] MeijerGForm(v1.2.…
$\text{Hypergeometric2F1}[a, b, c, 1-\frac{1}{z}] \left(\begin{cases} (1-z)^{-1+c} & \text{Abs}[z] < 1 \\ 0 & \text{True} \end{cases}\right)$	[▪] MeijerGForm(v1.2.…
$\text{Hypergeometric2F1}[a, b, c, 1-z] \left(\begin{cases} (1-z)^{-1+c} & \text{Abs}[z] < 1 \\ 0 & \text{True} \end{cases}\right)$	[▪] MeijerGForm(v1.2.…
$\begin{cases} (-1+z)^a & \text{Abs}[z] > 1 \\ 0 & \text{True} \end{cases}$	[▪] MeijerGForm(v1.2.…
$\text{AppellF3}[a, a1, b, b1, c, 1-\frac{1}{z}, 1-z] \left(\begin{cases} (-1+z)^{-1+c} & \text{Abs}[z] > 1 \\ 0 & \text{True} \end{cases}\right)$	[▪] MeijerGForm(v1.2.…
$\text{AppellF3}[a, a1, b, b1, c, 1-z, 1-\frac{1}{z}] \left(\begin{cases} (-1+z)^{-1+c} & \text{Abs}[z] > 1 \\ 0 & \text{True} \end{cases}\right)$	[▪] MeijerGForm(v1.2.…
$\text{Hypergeometric2F1}[a, b, c, 1-\frac{1}{z}] \left(\begin{cases} (-1+z)^{-1+c} & \text{Abs}[z] > 1 \\ 0 & \text{True} \end{cases}\right)$	[▪] MeijerGForm(v1.2.…
$\text{Hypergeometric2F1}[a, b, c, 1-z] \left(\begin{cases} (-1+z)^{-1+c} & \text{Abs}[z] > 1 \\ 0 & \text{True} \end{cases}\right)$	[▪] MeijerGForm(v1.2.…
PolyLog[2, z]	−MeijerG[{{1, 1, 1…
PolyLog[a, z]	−MeijerG[{Table[… if $a \in \mathbb{Z}$ && $a > 0$
ScorerGi[z]	$\frac{\text{MeijerG}[\{\{\frac{2}{3}\},\{\frac{1}{6}\}\},\{\{0,\frac{1}{3},\frac{2}{3}\}…}{2 \times 3^{1/6} \pi}$

ScorerGiPrime[z]	$-\frac{3^{1/6}\,\text{MeijerG}[\{\{\frac{1}{3}\},\{-\frac{1}{6}\}\},}{2\pi}$
ScorerHi[z]	$\frac{2\pi\,\text{MeijerG}[\{\{\frac{2}{3}\},\{\frac{1}{6},\frac{1}{2},\frac{5}{6}\}\},\{}{3^{1/6}}$
ScorerHiPrime[z]	$-2\times 3^{1/6}\,\pi\,\text{MeijerG}[$
$\text{Abs}[1-z]^{a}\,\text{Sign}[1-z]$	[•] MeijerGForm(v1.2.
$\text{Abs}[1-z]^{a}\,\text{Sign}[-1+z]$	[•] MeijerGForm(v1.2.
Sin[z]	$\sqrt{\pi}\,\text{MeijerG}[\{\{\},\{$
Sinc[z]	$\frac{1}{2}\sqrt{\pi}\,\text{MeijerG}[\{\{\},$
Sinh[z]	$-i\sqrt{\pi}\,\text{MeijerG}[\{\{$
SinhIntegral[z]	$\frac{1}{2}\pi^{3/2}\,\text{MeijerG}[\{\{\},$
SinIntegral[z]	$\frac{1}{2}\sqrt{\pi}\,\text{MeijerG}[\{\{1\}$
SphericalBesselJ[a, z]	$\frac{1}{2}\sqrt{\pi}\,\text{MeijerG}[\{\{\},$
SphericalBesselY[a, z]	$\frac{1}{2}\sqrt{\pi}\,\text{MeijerG}[\{\{\},$
SphericalHankelH1[a, z]	$\frac{1}{2}\sqrt{\pi}\,\text{MeijerG}[\{\{\},$
SphericalHankelH2[a, z]	$\frac{1}{2}\sqrt{\pi}\,\text{MeijerG}[\{\{\},$
StruveH[v, z]	$\text{MeijerG}[\{\{\frac{1+v}{2}\},\{\}$
StruveL[v, z]	$-\pi\,\text{Csc}[\frac{\pi v}{2}]\,\text{MeijerG}$
UnitStep[z]	MeijerG[{{}, {1}
UnitStep[1 − Abs[z]]	MeijerG[{{}, {1}}
$(1-z)^{a}$ UnitStep[1 − Abs[z]]	[•] MeijerGForm(v1.2.
$(1-z)^{-1+c}$ AppellF3[a, a1, b, b1, c, $1-\frac{1}{z}$, 1 − z] UnitStep[1 − Abs[z]]	Gamma[c] MeijerG
$(1-z)^{-1+c}$ AppellF3[a, a1, b, b1, c, 1 − z, $1-\frac{1}{z}$] UnitStep[1 − Abs[z]]	Gamma[c] MeijerG
UnitStep[−1 + Abs[z]]	MeijerG[{{1}, {}}
$(-1+z)^{a}$ UnitStep[−1 + Abs[z]]	[•] MeijerGForm(v1.2.
$(-1+z)^{-1+c}$ AppellF3[a, a1, b, b1, c, $1-\frac{1}{z}$, 1 − z] UnitStep[−1 + Abs[z]]	Gamma[c] MeijerG
$(-1+z)^{-1+c}$ AppellF3[a, a1, b, b1, c, 1 − z, $1-\frac{1}{z}$] UnitStep[−1 + Abs[z]]	Gamma[c] MeijerG
WeberE[v, z]	$\text{MeijerG}[\{\{0,\frac{1}{2}\},\{1$
WeberE[v, a, z]	$2^{a}\,\text{MeijerG}[\{\{-\frac{a}{2},\frac{1-}{2}$
WhittakerM[a, b, z]	$\sum_{k=0}^{-\frac{1}{2}+a-b} \frac{2^{\frac{1}{2}+k+b}\,\text{Pochham}}{}$ if $-\frac{1}{2}+a-b\in\mathbb{Z}$
$e^{-z/2}$ WhittakerM[a, b, z]	$\frac{\text{Gamma}[1+2b]\,\text{MeijerG}[\{\{}{\text{Gamma}}$

$e^{z/2}$ WhittakerM[a, b, z] $\frac{\pi\,\text{Gamma}[1+2b]\ \text{MeijerG}[\{}{\text{Ga}}$

WhittakerW[a, b, z] $(-1)^{-\frac{1}{2}+a-b}$ Pochha

$\sum_{k=0}^{-\frac{1}{2}+a-b} 2^{\frac{1}{2}+k+b}$ Poc

if $-\frac{1}{2}+a-b \in \mathbb{Z}$

$e^{-z/2}$ WhittakerW[a, b, z] MeijerG[{{}, {1 − a

$e^{z/2}$ WhittakerW[a, b, z] $\frac{\text{MeijerG}[\{\{1+a\},\{\}\},\{\{\frac{1}{2}+b}{\text{Gamma}[\frac{1}{2}-a-b]\ \text{Gamm}}$

In[∘]:=

```
Select[%, (FreeQ[#, MeijerG] ∨ Not[FreeQ[#, "MeijerGForm"]]) &]
```

Out[∘]=

{{BilateralHypergeometricPFQ[{a_1}, {b_1}, z],
  MeijerGForm(v1.2.0) [BilateralHypergeometricPFQ[{a_1}, {b_1}, z], z]},
 {BilateralHypergeometricPFQ[{a_1, a_2}, {b_1, b_2}, z],
  MeijerGForm(v1.2.0) [BilateralHypergeometricPFQ[{a_1, a_2}, {b_1, b_2}, z], z]},
 {BilateralHypergeometricPFQ[{a_1, a_2}, Table[b_i, {i, 1, q}], z],
  MeijerGForm(v1.2.0) [BilateralHypergeometricPFQ[{a_1, a_2}, Table[b_i, {i, 1, q}], z], z]},
 {BilateralHypergeometricPFQ[{a_1, a_2, a_3}, {b_1, b_2, b_3}, z],
  MeijerGForm(v1.2.0) [BilateralHypergeometricPFQ[{a_1, a_2, a_3}, {b_1, b_2, b_3}, z], z]},
 {BilateralHypergeometricPFQ[Table[a_i, {i, 1, p}], {b1, b2, b3}, z], MeijerGForm(v1.2.0) [
   BilateralHypergeometricPFQ[Table[a_i, {i, 1, p}], {b1, b2, b3}, z], z]},
 {BilateralHypergeometricPFQ[Table[a_i, {i, 1, p}], Table[b_i, {i, 1, q}], z],
  MeijerGForm(v1.2.0) [BilateralHypergeometricPFQ[Table[a_i, {i, 1, p}],
    Table[b_i, {i, 1, q}], z], z]}, {$(1-z)^a$ HeavisideTheta[1 − Abs[z]],
  MeijerGForm(v1.2.0) [$(1-z)^a$ HeavisideTheta[1 − Abs[z]], z]},
 {$(-1+z)^a$ HeavisideTheta[−1 + Abs[z]],
  MeijerGForm(v1.2.0) [$(-1+z)^a$ HeavisideTheta[−1 + Abs[z]], z]},
 {HermiteH[a, z], MeijerGForm(v1.2.0) [HermiteH[a, z], z]},
 {$\begin{cases} 1 & \text{Abs}[z] > 1 \\ 0 & \text{True} \end{cases}$, MeijerGForm(v1.2.0) [$\begin{cases} 1 & \text{Abs}[z] > 1 \\ 0 & \text{True} \end{cases}$, z]},
 {$\begin{cases} 1 & \text{Abs}[z] < 1 \\ 0 & \text{True} \end{cases}$, MeijerGForm(v1.2.0) [$\begin{cases} 1 & \text{Abs}[z] < 1 \\ 0 & \text{True} \end{cases}$, z]},

$$\left\{\left\{\begin{array}{ll}(1-z)^a & \text{Abs}[z]<1\\ 0 & \text{True}\end{array}\right.,\ \text{MeijerGForm}(\text{v1.2.0})\left[\left\{\begin{array}{ll}(1-z)^a & \text{Abs}[z]<1\\ 0 & \text{True}\end{array}\right., z\right]\right\},$$

$$\left\{\text{AppellF3}\left[a, a1, b, b1, c, 1-\frac{1}{z}, 1-z\right]\left(\left\{\begin{array}{ll}(1-z)^{-1+c} & \text{Abs}[z]<1\\ 0 & \text{True}\end{array}\right.\right),\right.$$
$$\left.\text{MeijerGForm}(\text{v1.2.0})\left[\text{AppellF3}\left[a, a1, b, b1, c, 1-\frac{1}{z}, 1-z\right]\left(\left\{\begin{array}{ll}(1-z)^{-1+c} & \text{Abs}[z]<1\\ 0 & \text{True}\end{array}\right.\right), z\right]\right\},$$

$$\left\{\text{AppellF3}\left[a, a1, b, b1, c, 1-z, 1-\frac{1}{z}\right]\left(\left\{\begin{array}{ll}(1-z)^{-1+c} & \text{Abs}[z]<1\\ 0 & \text{True}\end{array}\right.\right),\right.$$
$$\left.\text{MeijerGForm}(\text{v1.2.0})\left[\text{AppellF3}\left[a, a1, b, b1, c, 1-z, 1-\frac{1}{z}\right]\left(\left\{\begin{array}{ll}(1-z)^{-1+c} & \text{Abs}[z]<1\\ 0 & \text{True}\end{array}\right.\right), z\right]\right\},$$

$$\left\{\text{Hypergeometric2F1}\left[a, b, c, 1-\frac{1}{z}\right]\left(\left\{\begin{array}{ll}(1-z)^{-1+c} & \text{Abs}[z]<1\\ 0 & \text{True}\end{array}\right.\right),\right.$$
$$\left.\text{MeijerGForm}(\text{v1.2.0})\left[\text{Hypergeometric2F1}\left[a, b, c, 1-\frac{1}{z}\right]\left(\left\{\begin{array}{ll}(1-z)^{-1+c} & \text{Abs}[z]<1\\ 0 & \text{True}\end{array}\right.\right), z\right]\right\},$$

$$\left\{\text{Hypergeometric2F1}[a, b, c, 1-z]\left(\left\{\begin{array}{ll}(1-z)^{-1+c} & \text{Abs}[z]<1\\ 0 & \text{True}\end{array}\right.\right),\right.$$
$$\left.\text{MeijerGForm}(\text{v1.2.0})\left[\text{Hypergeometric2F1}[a, b, c, 1-z]\left(\left\{\begin{array}{ll}(1-z)^{-1+c} & \text{Abs}[z]<1\\ 0 & \text{True}\end{array}\right.\right), z\right]\right\},$$

$$\left\{\left\{\begin{array}{ll}(-1+z)^a & \text{Abs}[z]>1\\ 0 & \text{True}\end{array}\right.,\ \text{MeijerGForm}(\text{v1.2.0})\left[\left\{\begin{array}{ll}(-1+z)^a & \text{Abs}[z]>1\\ 0 & \text{True}\end{array}\right., z\right]\right\},$$

$$\left\{\text{AppellF3}\left[a, a1, b, b1, c, 1-\frac{1}{z}, 1-z\right]\left(\left\{\begin{array}{ll}(-1+z)^{-1+c} & \text{Abs}[z]>1\\ 0 & \text{True}\end{array}\right.\right), \text{MeijerGForm}(\text{v1.2.0})\right[$$
$$\left.\left.\text{AppellF3}\left[a, a1, b, b1, c, 1-\frac{1}{z}, 1-z\right]\left(\left\{\begin{array}{ll}(-1+z)^{-1+c} & \text{Abs}[z]>1\\ 0 & \text{True}\end{array}\right.\right), z\right]\right\},$$

$$\left\{\text{AppellF3}\left[a, a1, b, b1, c, 1-z, 1-\frac{1}{z}\right]\left(\left\{\begin{array}{ll}(-1+z)^{-1+c} & \text{Abs}[z]>1\\ 0 & \text{True}\end{array}\right.\right), \text{MeijerGForm}(\text{v1.2.0})\right[$$
$$\left.\left.\text{AppellF3}\left[a, a1, b, b1, c, 1-z, 1-\frac{1}{z}\right]\left(\left\{\begin{array}{ll}(-1+z)^{-1+c} & \text{Abs}[z]>1\\ 0 & \text{True}\end{array}\right.\right), z\right]\right\},$$

$$\left\{\text{Hypergeometric2F1}\left[a, b, c, 1-\frac{1}{z}\right]\left(\left\{\begin{array}{ll}(-1+z)^{-1+c} & \text{Abs}[z]>1\\ 0 & \text{True}\end{array}\right.\right),\right.$$
$$\left.\text{MeijerGForm}(\text{v1.2.0})\left[\text{Hypergeometric2F1}\left[a, b, c, 1-\frac{1}{z}\right]\left(\left\{\begin{array}{ll}(-1+z)^{-1+c} & \text{Abs}[z]>1\\ 0 & \text{True}\end{array}\right.\right), z\right]\right\},$$

$$\left\{\text{Hypergeometric2F1}[a, b, c, 1-z]\left(\left\{\begin{array}{ll}(-1+z)^{-1+c} & \text{Abs}[z]>1\\ 0 & \text{True}\end{array}\right.\right),\right.$$
$$\left.\text{MeijerGForm}(\text{v1.2.0})\left[\text{Hypergeometric2F1}[a, b, c, 1-z]\left(\left\{\begin{array}{ll}(-1+z)^{-1+c} & \text{Abs}[z]>1\\ 0 & \text{True}\end{array}\right.\right), z\right]\right\},$$

$$\left\{\text{Abs}[1-z]^a\,\text{Sign}[1-z],\ \text{MeijerGForm}(\text{v1.2.0})\left[\text{Abs}[1-z]^a\,\text{Sign}[1-z], z\right]\right\},$$

$$\left\{\text{Abs}[1-z]^a\,\text{Sign}[-1+z],\ \text{MeijerGForm}(\text{v1.2.0})\left[\text{Abs}[1-z]^a\,\text{Sign}[-1+z], z\right]\right\},$$

{(1 - z)^a UnitStep[1 - Abs[z]], MeijerGForm(v1.2.0)[(1 - z)^a UnitStep[1 - Abs[z]], z]},
{(-1 + z)^a UnitStep[-1 + Abs[z]],
MeijerGForm(v1.2.0)[(-1 + z)^a UnitStep[-1 + Abs[z]], z]}}

In[∘]:=

```
% // Length
```

Out[∘]=

25

Test of FoxHForm[FoxMeijer166]

```
{#, ResourceFunction["FoxHForm"][#, z]} & /@
   Table[FoxMeijer166⟦k⟧, {k, 1, 166}] // EchoTiming // TableForm (*2.44 Sec*)
```

2.44832

Out[∘]//TableForm=

a^z	FoxH[{{}, {}}, {·
e^z	FoxH[{{}, {}}, {·
$\frac{1}{1-z}$	π FoxH[{{{0, 1}},
$\sqrt{z}$	FoxH[{{}, {}}, {{
z^b	FoxH[{{}, {}}, {·
$(1+z)^a$	$\sum_{k=0}^{a} \frac{1}{k!}$ Pochhamr … (FoxH[{{}… {{{1… FoxH[{{{1+a,1}},{}} … Gamma[
Abs[1 - z]^a	π Sec[$\frac{a\pi}{2}$] FoxH[{{{1+a,1 …
AiryAi[z]	FoxH[{{},{}},{{{0,$\frac{1}{3}$},{$\frac{1}{3}$,… / $6\times3^{1/6}\pi$
AiryAiPrime[z]	− FoxH[{{},{}},{{{0,$\frac{1}{3}$},{… / $2\times3^{5/6}\pi$
AiryBi[z]	2 π FoxH[{{},{{$\frac{1}{6}$,$\frac{1}{3}$},{$\frac{2}{3}$,$\frac{1}{3}$}} …
AiryBiPrime[z]	− 2 π FoxH[{{},{{−$\frac{1}{6}$,$\frac{1}{3}$},{$\frac{1}{3}$ …
AngerJ[a, z]	$\frac{1}{2}$ FoxH[{{{0, $\frac{1}{2}$}, {$\frac{1}{2}$ …
AngerJ[a, b, z]	2^{-1+b} FoxH[{{{−$\frac{b}{2}$, …
ArcCos[z]	$\frac{1}{2}$ π (FoxH[{{}, {} …

ArcCosh[z]	$\frac{\pi\sqrt{-1+z}\ (\text{FoxH}[\{\{\},\{\},\ldots}{}$
ArcCot[z]	$\frac{1}{2}\, i\ (\text{FoxH}[\{\{\{1,1\}$
ArcCoth[z]	$\frac{1}{2}\ (-\text{FoxH}[\{\{\{1,1\}$
ArcCsc[z]	$-\frac{i\ \text{FoxH}[\{\{\{1,\frac{1}{2}\},\{1,\frac{1}{2}\}\},\{\}}{4\sqrt{}}$
ArcCsch[z]	$\frac{\text{FoxH}[\{\{\{1,\frac{1}{2}\},\{1,\frac{1}{2}\}\},\{\},\{\{}{4\sqrt{\pi}}$
ArcSec[z]	$\frac{1}{2}\pi\ (\text{FoxH}[\{\{\},\{\}$
ArcSech[z]	$\frac{\pi\sqrt{-1+\frac{1}{z}}\ (\text{FoxH}[\{\{\},\{\},}{}$
ArcSin[z]	$-\frac{i\ \text{FoxH}[\{\{\{1,\frac{1}{2}\},\{1,\frac{1}{2}\}\},\{\}}{4}$
ArcSinh[z]	$\frac{\text{FoxH}[\{\{\{1,\frac{1}{2}\},\{1,\frac{1}{2}\}\},\{\},\{\{}{4\sqrt{\pi}}$
ArcTan[z]	$\frac{1}{4}\text{FoxH}[\{\{\{\frac{1}{2},\frac{1}{2}\},\{1$
ArcTan[a, z]	(ArcTan[a, z] + i L
ArcTan[z, a]	(ArcTan[z, a] + i L
ArcTanh[z]	$-\frac{1}{4}\, i\ \text{FoxH}[\{\{\{\frac{1}{2},\frac{1}{2}\}$
BesselI[a, z]	$\frac{1}{2}\pi\ \text{FoxH}[\{\},\{\{\frac{1+}{2}$
BesselJ[a, z]	$\frac{1}{2}\text{FoxH}[\{\{\},\{\}\},\{$
BesselK[a, z]	$\frac{1}{4}\text{FoxH}[\{\{\},\{\}\},\{$
BesselY[a, z]	$\frac{1}{2}\text{FoxH}[\{\},\{\{\frac{1}{2}\ (-$
Beta[z, a, b]	$\frac{(-z)^{-a} z^{a}\ \text{FoxH}[\{\{\{1,1\},\{}{\text{G}}$
Beta[c, z, a, b]	−Beta[c, a, b] (Fo
Beta[z, c, a, b]	Beta[c, a, b] (FoxH
BetaRegularized[z, a, b]	$\frac{(-z)^{-a} z^{a}\ \text{FoxH}[\{\{\{1,1\},\{}{\text{Beta}[a}$
BetaRegularized[c, z, a, b]	$\frac{-\text{Beta}[c,a,b]\ (\text{FoxH}[\{\{\},\{}{}$
BetaRegularized[z, c, a, b]	$\frac{\text{Beta}[c,a,b]\ (\text{FoxH}[\{\{\},\{\}}{}$
BilateralHypergeometricPFQ[$\{a_1\}$, $\{b_1\}$, z]	FoxHForm (v1.0.0)
BilateralHypergeometricPFQ[$\{a_1, a_2\}$, $\{b_1, b_2\}$, z]	FoxHForm (v1.0.0)
BilateralHypergeometricPFQ[$\{a_1, a_2\}$, Table[b_i, {i, 1, q}], z]	FoxHForm (v1.0.0)
BilateralHypergeometricPFQ[$\{a_1, a_2, a_3\}$, $\{b_1, b_2, b_3\}$, z]	FoxHForm (v1.0.0)
BilateralHypergeometricPFQ[Table[a_i, {i, 1, p}], {b1, b2, b3}, z]	FoxHForm (v1.0.0)

BilateralHypergeometricPFQ[Table[a_i, {i, 1, p}], Table[b_i, {i, 1, q}], z]	FoxHForm (v1.0.0)
CarlsonRC[x, z]	$\frac{\text{FoxH}[\{\{\{0,1\},\{\frac{1}{2},1\}\},\{\},\{ }{2\sqrt{\pi}}$
CarlsonRC[z, y]	$\frac{\text{FoxH}[\{\{\{\frac{1}{2},1\},\{\frac{1}{2},1\}\},\{\},\{\{ }{2\sqrt{\pi}\ \sqrt{}}$
CarlsonRE[y, z]	$-\frac{\sqrt{y}\ \text{FoxH}[\{\{\{\frac{3}{2},1\},\{\frac{1}{2},1\}\},}{}$
CarlsonRK[y, z]	$\frac{\text{FoxH}[\{\{\{\frac{1}{2},1\},\{\frac{1}{2},1\}\},\{\},\{ }{\pi^2\sqrt{y}}$
ChebyshevT[a, z]	$\sum_{k=0}^{a}\frac{1}{k!}(-1)^k\sqrt{}$ … Hypergeom … (FoxH[{{ … {{{ … $-\frac{a\,\text{Sin}[a\,\pi]\ \text{FoxH}[\{\{\{}{}$
ChebyshevU[a, z]	$\sum_{k=0}^{a}\frac{1}{2\,k!}(-1)^k($ … Hypergeom … $\frac{1}{2}$] (FoxH … {{{ … $-\frac{\text{Sin}[a\,\pi]\ \text{FoxH}[\{\{\{1+}{}$
Cos[z]	$\frac{1}{2}\sqrt{\pi}$ FoxH[{ { }, {
Cosh[z]	$\frac{1}{2}\sqrt{\pi}$ FoxH[{ { }, {
CoshIntegral[z]	$-\frac{1}{4}\pi^{3/2}$ FoxH[{ { },
CosIntegral[z]	$-\frac{1}{4}\sqrt{\pi}$ FoxH[{ { }, {
DawsonF[z]	$\frac{1}{4}\sqrt{\pi}$ FoxH[{{{$\frac{1}{2}$, $\frac{1}{2}$
EllipticE[z]	$-\frac{1}{4}$ FoxH[{{{$\frac{1}{2}$, 1}, {
EllipticK[z]	$\frac{1}{2}$ FoxH[{{{$\frac{1}{2}$, 1}, {$\frac{1}{2}$
Erf[z]	$\frac{\text{FoxH}[\{\{\{1,\frac{1}{2}\}\},\{\}\},\{\{\{\frac{1}{2},\frac{1}{2}\}}{2\sqrt{\pi}}$
Erf[a, z]	$-\frac{\text{FoxH}[\{\{\{1,\frac{1}{2}\}\},\{\}\},\{\{\{\frac{1}{2},}{2\sqrt{\pi}}$
Erf[z, b]	$\frac{\text{FoxH}[\{\{\{1,\frac{1}{2}\}\},\{\}\},\{\{\{\frac{1}{2},\frac{1}{2}\}}{2\sqrt{\pi}}$
Erfc[z]	$\frac{\text{FoxH}[\{\{\},\{\{1,\frac{1}{2}\}\}\},\{\{\{0,\frac{1}{2}\}}{2\sqrt{\pi}}$
Erfi[z]	$-\frac{i\ \text{FoxH}[\{\{\{1,\frac{1}{2}\}\},\{\}\},\{\{\{}{2\sqrt{\pi}}$
ExpIntegralE[a, z]	FoxH[{ { }, { {a, 1}

ExpIntegralEi[z]	−FoxH[{{}, {{1,
Fibonacci[z]	2 FoxH[{{},{}},{{{0,1}}
Fibonacci[a, z]	$\sum_{k=0}^{\text{Floor}[\frac{1}{2}(-1+a)]}$ Bin … (FoxH[{{}, … FoxH[{{ … {{{−2 … Sin[a π] FoxH[{{{$\frac{1+a}{2}$,
FresnelC[z]	$e^{-\frac{i\pi}{4}}$ π FoxH[{{},{{1,$\frac{1}{4}$}}},{
FresnelF[z]	FoxH[{{{$\frac{3}{4}$,$\frac{1}{4}$}},{}},{{{0,$\frac{1}{4}$} / 8 $\sqrt{2}$ π
FresnelG[z]	FoxH[{{{$\frac{1}{4}$,$\frac{1}{4}$}},{}},{{{0,$\frac{1}{4}$} / 8 $\sqrt{2}$ π
FresnelS[z]	$e^{-\frac{3i\pi}{4}}$ π FoxH[{{},{{1,$\frac{1}{4}$}}},
Gamma[a, z]	FoxH[{{}, {{1, 1}
Gamma[a, b, z]	−FoxH[{{}, {{1, 1
Gamma[a, z, b]	FoxH[{{}, {{1, 1}
GammaRegularized[a, z]	FoxH[{{},{{1,1}}},{{{0, / Gamma[a
GammaRegularized[a, b, z]	− FoxH[{{},{{1,1}}},{{{ / Gamma[
GammaRegularized[a, z, b]	FoxH[{{},{{1,1}}},{{{0, / Gamma[a
GegenbauerC[a, b, z]	$\sum_{k=0}^{a} \frac{1}{k!\,a!\,\text{Gamma}[b]}$ … Gamma[a … {−a, 1, a … (FoxH[{{ … {{{ … $\sum_{k=0}^{\infty} \frac{1}{k!\,\text{Gamma}[1+a]}$ … Gamma[a … {−a, 1, a … (FoxH[{{ … {{{
HankelH1[a, z]	$\frac{1}{2}$ FoxH[{{}, {}}, {
HankelH2[a, z]	$\frac{1}{2}$ FoxH[{{}, {}}, {
Haversine[z]	$\frac{1}{4}$ $\sqrt{\pi}$ FoxH[{{{1, $\frac{1}{2}$

(1 – z) ^a HeavisideTheta[1 – Abs[z]]	[▪] FoxHForm (v1.0.0)
(–1 + z) ^a HeavisideTheta[–1 + Abs[z]]	[▪] FoxHForm (v1.0.0)
HermiteH[a, z]	[▪] FoxHForm (v1.0.0)
Hypergeometric0F1[a, z]	Gamma[a] FoxH[{
Hypergeometric0F1Regularized[a, z]	FoxH[{{}, {}}, {
Hypergeometric1F1[a, b, z]	π Gamma[b] FoxH[{{{1–a
Hypergeometric1F1Regularized[a, b, z]	π FoxH[{{{1–a,1}},{{$\frac{1}{2}$,1} Ga
Hypergeometric2F1[a, b, c, z]	Gamma[c] FoxH[{{{1–a,1 Ga
Hypergeometric2F1Regularized[a, b, c, z]	FoxH[{{{1–a,1},{1–b,1}} Gamma[
HypergeometricPFQ[{a_1, a_2}, Table[b_j, {j, 1, q}], z]	FoxH[{{{1–a_1,1},{1–a_2,1}
HypergeometricPFQ[Table[a_j, {j, 1, p}], {b1, b2, b3}, z]	Gamma[b1] Gamma[b2] G
HypergeometricPFQ[Table[a_j, {j, 1, p}], Table[b_j, {j, 1, q}], z]	FoxH[{Table[{1–a_j,1},{{j,1
HypergeometricPFQRegularized[{a_1, a_2}, Table[b_j, {j, 1, q}], z]	FoxH[{{{1–a_1,1},{1–a_2,1} G
HypergeometricPFQRegularized[Table[a_j, {j, 1, p}], {b1, b2, b3}, z]	FoxH[{Table[{1–a_j,1},{{j,1
HypergeometricPFQRegularized[Table[a_j, {j, 1, p}], Table[b_j, {j, 1, q}], z]	FoxH[{Table[{1–a_j,1},{{j,1
HypergeometricU[a, b, z]	FoxH[{{{1–a,1}},{}},{{ Gamma[a] Gam
InverseHaversine[z]	– $\sqrt{-z}$ FoxH[{{{1,1},{1,1
KelvinBei[0, z]	$\frac{1}{4}$ π FoxH[{{}, {}}
KelvinBei[a, z]	$\frac{1}{4}$ π FoxH[{{}, {{a,
KelvinBer[0, z]	$\frac{1}{4}$ π FoxH[{{}, {}}
KelvinBer[a, z]	$\frac{1}{4}$ π FoxH[{{}, {{$\frac{1}{2}$
KelvinKei[0, z]	–$\frac{1}{16}$ FoxH[{{}, {}}
KelvinKei[a, z]	–$\frac{1}{16}$ FoxH[{{}, {{$\frac{a}{2}$,
KelvinKer[0, z]	$\frac{1}{16}$ FoxH[{{}, {}},
KelvinKer[a, z]	$\frac{1}{16}$ FoxH[{{}, {{$\frac{1+a}{2}$
LaguerreL[a, z]	$\sum_{k=0}^{a} \frac{1}{(k!)^2}$ Pochha { (FoxH[{{}, {{{1 – Gamma[1 + a] FoxH[{{}, {{

LaguerreL[a, b, z]	$\sum_{k=0}^{a} \frac{1}{k!\, a!\, \text{Gamma}[1+}$ … (FoxH[{{}, … {{{1 … Gamma[1 + a + b … FoxH[{{}, {{ …
LegendreP[v, z]	$\sum_{k=0}^{\text{Floor}[\frac{v}{2}]} (-1)^{k}\, 2^{}$ … $\sum_{k=0}^{\infty} \frac{1}{k!} (-1)^{k}$ Hy … (FoxH[{{ … {{{ … ! ((v ∈ ℤ && v ≥ …
LegendreP[a, b, z]	$-\frac{(1-z)^{-b/2} (1+z)^{b/2} \text{Sin}[}{}$ …
LegendreP[a, b, 2, z]	$\sum_{k=0}^{a} (-1)^{k}$ Gamr … (FoxH[{{}, … {{{1 … $\sum_{j=0}^{k} \frac{1}{j!\,(-j+k)!}$ H … {1 − b, … $\sum_{k=0}^{\infty} (-1)^{k}$ Gamr … (FoxH[{{}, … {{{1 … $\sum_{j=0}^{k} \frac{1}{j!\,(-j+k)!}$ H … {1 − b, …
LegendreP[a, b, 3, z]	$-\frac{(-1+z)^{-b/2} (1+z)^{b/2} \text{Sin}}{}$ …
LegendreQ[v, z]	$\frac{1}{2}$ (FoxH[{{{1 + v, … Cos[π v] F … {{{0, …
LegendreQ[a, b, z]	$-\frac{1}{2}$ Csc[b π] Sin[a … {{{1 + … $(1-z)^{-b/2}$ … {{{0, …

LegendreQ[a, b, 2, z]	$-\frac{1}{2}$ Csc[b π] Sin[a {{{1 + $(1-z)^{-b/2}$ {{{0,
LegendreQ[a, b, 3, z]	$-\frac{1}{2}\, e^{i\, b\, \pi}$ Csc[b π] $(-(-1+z))^{b/2}$ {{{1 + $(-1+z)^{-b/}$ {{{0,
Log[z]	− FoxH[{{{0, 1}, {
LucasL[z]	$\frac{1}{2}$ (2 FoxH[{{}, {}

LucasL[a, z]	$\sum_{k=0}^{\text{Floor}\left[\frac{a}{2}\right]} \frac{1}{k!}\, 2^{1+2k-}$
	Hypergeom
	(FoxH[{{
	FoxH[
	{{{
	− a Sin[a π] FoxH[{{{

ParabolicCylinderD[a, z]	$(-1)^{\text{Floor}\left[\frac{a}{2}\right]}\, 2^{\text{Floor}\left[\frac{1}{}\right.}$ $\sum_{k=0}^{\text{Floor}\left[\frac{a}{2}\right]} \frac{2^{-1+k}\ \text{Poch}}{}$ if a ∈ ℤ && a ≥ 0
$\begin{cases} 1 & \text{Abs[z]} > 1 \\ 0 & \text{True} \end{cases}$	FoxHForm (v1.0.0)
$\begin{cases} 1 & \text{Abs[z]} < 1 \\ 0 & \text{True} \end{cases}$	FoxHForm (v1.0.0)
$\begin{cases} (1-z)^{a} & \text{Abs[z]} < 1 \\ 0 & \text{True} \end{cases}$	FoxHForm (v1.0.0)
AppellF3$\left[a, a1, b, b1, c, 1-\frac{1}{z}, 1-z\right] \left(\begin{cases} (1-z)^{-1+c} & \text{Abs[z]} < 1 \\ 0 & \text{True} \end{cases}\right)$	FoxHForm (v1.0.0)
AppellF3$\left[a, a1, b, b1, c, 1-z, 1-\frac{1}{z}\right] \left(\begin{cases} (1-z)^{-1+c} & \text{Abs[z]} < 1 \\ 0 & \text{True} \end{cases}\right)$	FoxHForm (v1.0.0)
Hypergeometric2F1$\left[a, b, c, 1-\frac{1}{z}\right] \left(\begin{cases} (1-z)^{-1+c} & \text{Abs[z]} < 1 \\ 0 & \text{True} \end{cases}\right)$	FoxHForm (v1.0.0)
Hypergeometric2F1$[a, b, c, 1-z] \left(\begin{cases} (1-z)^{-1+c} & \text{Abs[z]} < 1 \\ 0 & \text{True} \end{cases}\right)$	FoxHForm (v1.0.0)

$\begin{cases} (-1+z)^{a} & \text{Abs}[z]>1 \\ 0 & \text{True} \end{cases}$	[•] FoxHForm (v1.0.0)
$\text{AppellF3}\left[a, a1, b, b1, c, 1-\frac{1}{z}, 1-z\right]\left(\begin{cases} (-1+z)^{-1+c} & \text{Abs}[z]>1 \\ 0 & \text{True} \end{cases}\right)$	[•] FoxHForm (v1.0.0)
$\text{AppellF3}\left[a, a1, b, b1, c, 1-z, 1-\frac{1}{z}\right]\left(\begin{cases} (-1+z)^{-1+c} & \text{Abs}[z]>1 \\ 0 & \text{True} \end{cases}\right)$	[•] FoxHForm (v1.0.0)
$\text{Hypergeometric2F1}\left[a, b, c, 1-\frac{1}{z}\right]\left(\begin{cases} (-1+z)^{-1+c} & \text{Abs}[z]>1 \\ 0 & \text{True} \end{cases}\right)$	[•] FoxHForm (v1.0.0)
$\text{Hypergeometric2F1}[a, b, c, 1-z]\left(\begin{cases} (-1+z)^{-1+c} & \text{Abs}[z]>1 \\ 0 & \text{True} \end{cases}\right)$	[•] FoxHForm (v1.0.0)
PolyLog[2, z]	−FoxH[{{{1, 1}, {
PolyLog[a, z]	−FoxH[{Table[{1 {{{1, 1}}, T
ScorerGi[z]	$\frac{\text{FoxH}[\{\{\{\frac{2}{3},\frac{1}{3}\}\},\{\{\frac{1}{6},\frac{1}{3}\}\}\},\{\{}{6}$
ScorerGiPrime[z]	$-\frac{\text{FoxH}[\{\{\{\frac{1}{3},\frac{1}{3}\}\},\{\{-\frac{1}{6},\frac{1}{3}\}\}}{}$
ScorerHi[z]	$\frac{2\pi\,\text{FoxH}[\{\{\{\frac{2}{3},\frac{1}{3}\}\},\{\{\frac{1}{6},\frac{1}{3}\},\{}{}$
ScorerHiPrime[z]	$-\frac{2\pi\,\text{FoxH}[\{\{\{\frac{1}{3},\frac{1}{3}\}\},\{\{-\frac{1}{6},}{}$
$\text{Abs}[1-z]^{a}\,\text{Sign}[1-z]$	[•] FoxHForm (v1.0.0)
$\text{Abs}[1-z]^{a}\,\text{Sign}[-1+z]$	[•] FoxHForm (v1.0.0)
Sin[z]	$\frac{1}{2}\sqrt{\pi}$ FoxH[{{}, {
Sinc[z]	$\frac{1}{4}\sqrt{\pi}$ FoxH[{{}, {
Sinh[z]	$-\frac{1}{2}\,i\,\sqrt{\pi}$ FoxH[{{}
SinhIntegral[z]	$\frac{1}{4}\pi^{3/2}$ FoxH[{{}, {{
SinIntegral[z]	$\frac{1}{4}\sqrt{\pi}$ FoxH[{{{1, $\frac{1}{2}$
SphericalBesselJ[a, z]	$\frac{1}{4}\sqrt{\pi}$ FoxH[{{}, {
SphericalBesselY[a, z]	$\frac{1}{4}\sqrt{\pi}$ FoxH[{{}, {{
SphericalHankelH1[a, z]	$\frac{1}{4}\sqrt{\pi}$ FoxH[{{}, {
SphericalHankelH2[a, z]	$\frac{1}{4}\sqrt{\pi}$ FoxH[{{}, {
StruveH[v, z]	$\frac{1}{2}$ FoxH[{{{$\frac{1+v}{2}$, $\frac{1}{2}$}}
StruveL[v, z]	$-\frac{1}{2}\pi\,\text{Csc}[\frac{\pi v}{2}]$ FoxH[
UnitStep[z]	FoxH[{{}, {{1, 1
UnitStep[1 − Abs[z]]	FoxH[{{}, {{1, 1}
$(1-z)^{a}$ UnitStep[1 − Abs[z]]	[•] FoxHForm (v1.0.0)
$(1-z)^{-1+c}\,\text{AppellF3}\left[a, a1, b, b1, c, 1-\frac{1}{z}, 1-z\right]$ UnitStep[1 − Abs[z]]	Gamma[c] FoxH[{

$(1-z)^{-1+c}$ AppellF3[a, a1, b, b1, c, 1 − z, 1 − $\frac{1}{z}$] UnitStep[1 − Abs[z]]	Gamma[c] FoxH[{
UnitStep[−1 + Abs[z]]	FoxH[{{{1, 1}}, {
$(-1+z)^{a}$ UnitStep[−1 + Abs[z]]	[•] FoxHForm (v1.0.0)
$(-1+z)^{-1+c}$ AppellF3[a, a1, b, b1, c, 1 − $\frac{1}{z}$, 1 − z] UnitStep[−1 + Abs[z]]	Gamma[c] FoxH[{
$(-1+z)^{-1+c}$ AppellF3[a, a1, b, b1, c, 1 − z, 1 − $\frac{1}{z}$] UnitStep[−1 + Abs[z]]	Gamma[c] FoxH[{
WeberE[ν, z]	$\frac{1}{2}$ FoxH[{{{0, $\frac{1}{2}$}, {$\frac{1}{2}$
WeberE[ν, a, z]	2^{-1+a} FoxH[{{{−$\frac{a}{2}$, $\frac{1}{2}$
WhittakerM[a, b, z]	$\sum_{k=0}^{-\frac{1}{2}+a-b}$ $2^{\frac{1}{2}+k+b}$ Pochham … if $-\frac{1}{2}+a-b \in \mathbb{Z}$
$e^{-z/2}$ WhittakerM[a, b, z]	Gamma[1+2 b] FoxH[{{{1 … / Ga
$e^{z/2}$ WhittakerM[a, b, z]	π Gamma[1+2 b] FoxH[{{
WhittakerW[a, b, z]	$(-1)^{-\frac{1}{2}+a-b}$ Pochha … $\sum_{k=0}^{-\frac{1}{2}+a-b}$ $2^{\frac{1}{2}+k+b}$ Poc … if $-\frac{1}{2}+a-b \in \mathbb{Z}$
$e^{-z/2}$ WhittakerW[a, b, z]	FoxH[{{}, {{1 − a,
$e^{z/2}$ WhittakerW[a, b, z]	FoxH[{{{1+a,1}},{}},{{{ … / Gamma[$\frac{1}{2}$−a−b] G

In[∘]:=

```
Select[%, (FreeQ[#, FoxH] ∨ Not[FreeQ[#, "FoxHForm"]]) &]
```

Out[∘]=

{{BilateralHypergeometricPFQ[{a_1}, {b_1}, z],
[•] FoxHForm (v1.0.0) [BilateralHypergeometricPFQ[{a_1}, {b_1}, z], z]},
{BilateralHypergeometricPFQ[{a_1, a_2}, {b_1, b_2}, z],
[•] FoxHForm (v1.0.0) [BilateralHypergeometricPFQ[{a_1, a_2}, {b_1, b_2}, z], z]},
{BilateralHypergeometricPFQ[{a_1, a_2}, Table[b_i, {i, 1, q}], z],
[•] FoxHForm (v1.0.0) [BilateralHypergeometricPFQ[{a_1, a_2}, Table[b_i, {i, 1, q}], z], z]},
{BilateralHypergeometricPFQ[{a_1, a_2, a_3}, {b_1, b_2, b_3}, z],
[•] FoxHForm (v1.0.0) [BilateralHypergeometricPFQ[{a_1, a_2, a_3}, {b_1, b_2, b_3}, z], z]},
{BilateralHypergeometricPFQ[Table[a_i, {i, 1, p}], {b1, b2, b3}, z], [•] FoxHForm (v1.0.0) [

$$\text{BilateralHypergeometricPFQ}[\text{Table}[a_i, \{i, 1, p\}], \{b1, b2, b3\}, z], z]\Big\},$$

$$\Big\{\text{BilateralHypergeometricPFQ}[\text{Table}[a_i, \{i, 1, p\}], \text{Table}[b_i, \{i, 1, q\}], z],$$

$$\text{FoxHForm (v1.0.0)}[\text{BilateralHypergeometricPFQ}[\text{Table}[a_i, \{i, 1, p\}],$$

$$\text{Table}[b_i, \{i, 1, q\}], z], z]\Big\}, \Big\{(1-z)^a \,\text{HeavisideTheta}[1-\text{Abs}[z]],$$

$$\text{FoxHForm (v1.0.0)}[(1-z)^a \,\text{HeavisideTheta}[1-\text{Abs}[z]], z]\Big\},$$

$$\Big\{(-1+z)^a \,\text{HeavisideTheta}[-1+\text{Abs}[z]],$$

$$\text{FoxHForm (v1.0.0)}[(-1+z)^a \,\text{HeavisideTheta}[-1+\text{Abs}[z]], z]\Big\},$$

$$\Big\{\text{HermiteH}[a, z], \text{FoxHForm (v1.0.0)}[\text{HermiteH}[a, z], z]\Big\},$$

$$\left\{\begin{cases} 1 & \text{Abs}[z] > 1 \\ 0 & \text{True} \end{cases}, \text{FoxHForm (v1.0.0)}\left[\begin{cases} 1 & \text{Abs}[z] > 1 \\ 0 & \text{True} \end{cases}, z\right]\right\},$$

$$\left\{\begin{cases} 1 & \text{Abs}[z] < 1 \\ 0 & \text{True} \end{cases}, \text{FoxHForm (v1.0.0)}\left[\begin{cases} 1 & \text{Abs}[z] < 1 \\ 0 & \text{True} \end{cases}, z\right]\right\},$$

$$\left\{\begin{cases} (1-z)^a & \text{Abs}[z] < 1 \\ 0 & \text{True} \end{cases}, \text{FoxHForm (v1.0.0)}\left[\begin{cases} (1-z)^a & \text{Abs}[z] < 1 \\ 0 & \text{True} \end{cases}, z\right]\right\},$$

$$\left\{\text{AppellF3}\left[a, a1, b, b1, c, 1-\frac{1}{z}, 1-z\right]\left(\begin{cases} (1-z)^{-1+c} & \text{Abs}[z] < 1 \\ 0 & \text{True} \end{cases}\right),\right.$$

$$\left.\text{FoxHForm (v1.0.0)}\left[\text{AppellF3}\left[a, a1, b, b1, c, 1-\frac{1}{z}, 1-z\right]\left(\begin{cases} (1-z)^{-1+c} & \text{Abs}[z] < 1 \\ 0 & \text{True} \end{cases}\right), z\right]\right\},$$

$$\left\{\text{AppellF3}\left[a, a1, b, b1, c, 1-z, 1-\frac{1}{z}\right]\left(\begin{cases} (1-z)^{-1+c} & \text{Abs}[z] < 1 \\ 0 & \text{True} \end{cases}\right),\right.$$

$$\left.\text{FoxHForm (v1.0.0)}\left[\text{AppellF3}\left[a, a1, b, b1, c, 1-z, 1-\frac{1}{z}\right]\left(\begin{cases} (1-z)^{-1+c} & \text{Abs}[z] < 1 \\ 0 & \text{True} \end{cases}\right), z\right]\right\},$$

$$\left\{\text{Hypergeometric2F1}\left[a, b, c, 1-\frac{1}{z}\right]\left(\begin{cases} (1-z)^{-1+c} & \text{Abs}[z] < 1 \\ 0 & \text{True} \end{cases}\right),\right.$$

$$\left.\text{FoxHForm (v1.0.0)}\left[\text{Hypergeometric2F1}\left[a, b, c, 1-\frac{1}{z}\right]\left(\begin{cases} (1-z)^{-1+c} & \text{Abs}[z] < 1 \\ 0 & \text{True} \end{cases}\right), z\right]\right\},$$

$$\left\{\text{Hypergeometric2F1}[a, b, c, 1-z]\left(\begin{cases} (1-z)^{-1+c} & \text{Abs}[z] < 1 \\ 0 & \text{True} \end{cases}\right),\right.$$

$$\left.\text{FoxHForm (v1.0.0)}\left[\text{Hypergeometric2F1}[a, b, c, 1-z]\left(\begin{cases} (1-z)^{-1+c} & \text{Abs}[z] < 1 \\ 0 & \text{True} \end{cases}\right), z\right]\right\},$$

$$\left\{\begin{cases} (-1+z)^a & \text{Abs}[z] > 1 \\ 0 & \text{True} \end{cases}, \text{FoxHForm (v1.0.0)}\left[\begin{cases} (-1+z)^a & \text{Abs}[z] > 1 \\ 0 & \text{True} \end{cases}, z\right]\right\},$$

$$\left\{\text{AppellF3}\left[a, a1, b, b1, c, 1-\frac{1}{z}, 1-z\right]\left(\begin{cases} (-1+z)^{-1+c} & \text{Abs}[z] > 1 \\ 0 & \text{True} \end{cases}\right),\right.$$

$$\left.\text{FoxHForm (v1.0.0)}\left[\text{AppellF3}\left[a, a1, b, b1, c, 1-\frac{1}{z}, 1-z\right]\left(\begin{cases} (-1+z)^{-1+c} & \text{Abs}[z] > 1 \\ 0 & \text{True} \end{cases}\right), z\right]\right\},$$

$\left\{\text{AppellF3}\left[a, a1, b, b1, c, 1-z, 1-\frac{1}{z}\right]\left(\begin{cases}(-1+z)^{-1+c} & \text{Abs}[z]>1\\ 0 & \text{True}\end{cases}\right),\right.$

$\left.\text{FoxHForm}(\text{v1.0.0})\left[\text{AppellF3}\left[a, a1, b, b1, c, 1-z, 1-\frac{1}{z}\right]\left(\begin{cases}(-1+z)^{-1+c} & \text{Abs}[z]>1\\ 0 & \text{True}\end{cases}\right), z\right]\right\},$

$\left\{\text{Hypergeometric2F1}\left[a, b, c, 1-\frac{1}{z}\right]\left(\begin{cases}(-1+z)^{-1+c} & \text{Abs}[z]>1\\ 0 & \text{True}\end{cases}\right),\right.$

$\left.\text{FoxHForm}(\text{v1.0.0})\left[\text{Hypergeometric2F1}\left[a, b, c, 1-\frac{1}{z}\right]\left(\begin{cases}(-1+z)^{-1+c} & \text{Abs}[z]>1\\ 0 & \text{True}\end{cases}\right), z\right]\right\},$

$\left\{\text{Hypergeometric2F1}[a, b, c, 1-z]\left(\begin{cases}(-1+z)^{-1+c} & \text{Abs}[z]>1\\ 0 & \text{True}\end{cases}\right),\right.$

$\left.\text{FoxHForm}(\text{v1.0.0})\left[\text{Hypergeometric2F1}[a, b, c, 1-z]\left(\begin{cases}(-1+z)^{-1+c} & \text{Abs}[z]>1\\ 0 & \text{True}\end{cases}\right), z\right]\right\},$

$\left\{\text{Abs}[1-z]^a\,\text{Sign}[1-z], \text{FoxHForm}(\text{v1.0.0})[\text{Abs}[1-z]^a\,\text{Sign}[1-z], z]\right\},$

$\left\{\text{Abs}[1-z]^a\,\text{Sign}[-1+z], \text{FoxHForm}(\text{v1.0.0})[\text{Abs}[1-z]^a\,\text{Sign}[-1+z], z]\right\},$

$\left\{(1-z)^a\,\text{UnitStep}[1-\text{Abs}[z]], \text{FoxHForm}(\text{v1.0.0})[(1-z)^a\,\text{UnitStep}[1-\text{Abs}[z]], z]\right\},$

$\left\{(-1+z)^a\,\text{UnitStep}[-1+\text{Abs}[z]],\right.$

$\left.\text{FoxHForm}(\text{v1.0.0})[(-1+z)^a\,\text{UnitStep}[-1+\text{Abs}[z]], z]\right\}\}$

In[∘]:=

```
% // Length
```

Out[∘]=

25

```
NadList25 = Part[#, 1] & /@ %344
   (* this is the List25 which should be fixed or added into program *)
```

Out[•]=

$$\Big\{\text{BilateralHypergeometricPFQ}[\{a_1\},\{b_1\},z],\ \text{BilateralHypergeometricPFQ}[\{a_1,a_2\},\{b_1,b_2\},z],$$

$$\text{BilateralHypergeometricPFQ}[\{a_1,a_2\},\text{Table}[b_i,\{i,1,q\}],z],$$

$$\text{BilateralHypergeometricPFQ}[\{a_1,a_2,a_3\},\{b_1,b_2,b_3\},z],$$

$$\text{BilateralHypergeometricPFQ}[\text{Table}[a_i,\{i,1,p\}],\{b1,b2,b3\},z],$$

$$\text{BilateralHypergeometricPFQ}[\text{Table}[a_i,\{i,1,p\}],\text{Table}[b_i,\{i,1,q\}],z],$$

$$(1-z)^a\,\text{HeavisideTheta}[1-\text{Abs}[z]],\ (-1+z)^a\,\text{HeavisideTheta}[-1+\text{Abs}[z]],$$

$$\text{HermiteH}[a,z],\ \begin{cases}1 & \text{Abs}[z]>1\\ 0 & \text{True}\end{cases},\ \begin{cases}1 & \text{Abs}[z]<1\\ 0 & \text{True}\end{cases},\ \begin{cases}(1-z)^a & \text{Abs}[z]<1\\ 0 & \text{True}\end{cases},$$

$$\text{AppellF3}\Big[a,a1,b,b1,c,1-\frac{1}{z},1-z\Big]\left(\begin{cases}(1-z)^{-1+c} & \text{Abs}[z]<1\\ 0 & \text{True}\end{cases}\right),$$

$$\text{AppellF3}\Big[a,a1,b,b1,c,1-z,1-\frac{1}{z}\Big]\left(\begin{cases}(1-z)^{-1+c} & \text{Abs}[z]<1\\ 0 & \text{True}\end{cases}\right),$$

$$\text{Hypergeometric2F1}\Big[a,b,c,1-\frac{1}{z}\Big]\left(\begin{cases}(1-z)^{-1+c} & \text{Abs}[z]<1\\ 0 & \text{True}\end{cases}\right),$$

$$\text{Hypergeometric2F1}[a,b,c,1-z]\left(\begin{cases}(1-z)^{-1+c} & \text{Abs}[z]<1\\ 0 & \text{True}\end{cases}\right),\ \begin{cases}(-1+z)^a & \text{Abs}[z]>1\\ 0 & \text{True}\end{cases},$$

$$\text{AppellF3}\Big[a,a1,b,b1,c,1-\frac{1}{z},1-z\Big]\left(\begin{cases}(-1+z)^{-1+c} & \text{Abs}[z]>1\\ 0 & \text{True}\end{cases}\right),$$

$$\text{AppellF3}\Big[a,a1,b,b1,c,1-z,1-\frac{1}{z}\Big]\left(\begin{cases}(-1+z)^{-1+c} & \text{Abs}[z]>1\\ 0 & \text{True}\end{cases}\right),$$

$$\text{Hypergeometric2F1}\Big[a,b,c,1-\frac{1}{z}\Big]\left(\begin{cases}(-1+z)^{-1+c} & \text{Abs}[z]>1\\ 0 & \text{True}\end{cases}\right),$$

$$\text{Hypergeometric2F1}[a,b,c,1-z]\left(\begin{cases}(-1+z)^{-1+c} & \text{Abs}[z]>1\\ 0 & \text{True}\end{cases}\right),\ \text{Abs}[1-z]^a\,\text{Sign}[1-z],$$

$$\text{Abs}[1-z]^a\,\text{Sign}[-1+z],\ (1-z)^a\,\text{UnitStep}[1-\text{Abs}[z]],\ (-1+z)^a\,\text{UnitStep}[-1+\text{Abs}[z]]\Big\}$$

Case of MeijerG: PolyLog[2,z]

```
PolyLog[2, z] == z HypergeometricPFQ[{1, 1, 1}, {2, 2}, z]
```

```
PolyLog[2, z] == -MeijerG[{{1, 1, 1}, {}}, {{1}, {0, 0}}, -z]
```

$$\text{PolyLog}[2,z] == \sum_{k=1}^{\infty}\frac{z^k}{k^2}\ /;\ \text{Abs}[z]<1$$

$$\text{PolyLog}[2,z] == -\text{PolyLog}[2,1-z] + \frac{\pi^2}{6} - \text{Log}[z]\,\text{Log}[1-z]$$

$$\text{PolyLog}[2,z] == \frac{\pi^2}{6} - \sum_{k=1}^{\infty} \frac{(1-z)^k}{k^2} + \text{Log}[1-z] \sum_{k=1}^{\infty} \frac{(1-z)^k}{k}\ /;\ \text{Abs}[z-1] < 1$$

$$\text{PolyLog}[2,z] == \frac{\pi^2}{6} + (z-1)\ \text{HypergeometricPFQ}[\{1,1,1\},\{2,2\},1-z] - \text{Log}[1-z]\ (z-1)\ \text{Hypergeometric2F1}[1,1,2,1-z]$$

$$\text{PolyLog}[2,z] == -\frac{1}{2}\text{Log}[-z]^2 - \frac{\pi^2}{6} - \sum_{k=1}^{\infty} \frac{1}{k^2 z^k}\ /;\ \text{Abs}[z] > 1$$

$$\text{PolyLog}[2,z] == -\frac{1}{2}\text{Log}[-z]^2 - \frac{\pi^2}{6} - \text{PolyLog}\left[2,\frac{1}{z}\right]\ /;\ \text{Not}[\text{IntervalMemberQ}[\text{Interval}[\{0,1\}],z]]$$

$$\text{PolyLog}[2,z] == -\sum_{j=1}^{\infty} \text{Residue}\left[\frac{\text{Gamma}[-s]^3\ (-z)^{-s}}{\text{Gamma}[1-s]^2}\ \text{Gamma}[s+1],\{s,-j\}\right]\ /;\ \text{Abs}[z] < 1$$

Case of MeijerG: PolyLog[n,z]

$$\text{PolyLog}[n,z] == z\,\text{HypergeometricPFQ}[\{1,a_1,a_2,\ldots,a_n\},\{1+a_1,1+a_2,\ldots,1+a_n\},z]\ /;\ a_1 == a_2 == \ldots == a_n == 1 \land n \in \text{Integers} \land n > 0$$

$$\text{PolyLog}[n,z] == -\text{MeijerG}[\{\text{Table}[1,n+1],\{\}\},\{\{1\},\text{Table}[0,n]\},-z]\ /;\ n \in \text{Integers} \&\& n > 0$$

$$\text{PolyLog}[n,z] == \sum_{k=1}^{\infty} \frac{z^k}{k^n}\ /;\ \text{Abs}[z] < 1$$

$$\text{PolyLog}[n, z] == \frac{(z-1)^{n-1}\,(-\text{Log}[1-z] + \text{PolyGamma}[n] + \text{EulerGamma})}{(n-1)\,!}$$

$$\sum_{k=0}^{\infty}\Bigg(\text{KroneckerDelta}[k] + \frac{\text{UnitStep}[k-1]}{k\,!}\,\text{BellY}\Big[$$

$$\text{Table}\Big[\Big\{(-1)^{i}\,\text{Pochhammer}[1-n, i], \frac{i\,!\;(-1)^{i}}{i+1}\Big\}, \{i, k\}\Big]\Big]\Bigg)\,(z-1)^{k} -$$

$$\frac{(z-1)^{n}}{(n-1)\,!}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{u=0}^{j}\frac{(-1)^{j-u+1}\;(z-1)^{k}}{j-u+2}\Bigg(\text{KroneckerDelta}[k-j] + \frac{\text{UnitStep}[k-j-1]}{(k-j)\,!}$$

$$\text{BellY}\Big[\text{Table}\Big[\Big\{(-1)^{i}\,\text{Pochhammer}[1-n, i], \frac{i\,!\;(-1)^{i}}{i+1}\Big\}, \{i, k-j\}\Big]\Big]\Bigg)$$

$$\Bigg(\text{KroneckerDelta}[u] + \frac{\text{UnitStep}[u-1]}{u\,!}$$

$$\text{BellY}\Big[\text{Table}\Big[\Big\{\frac{i\,!\;(-1)^{i}}{i+1}, \frac{i\,!\;(-1)^{i}}{i+1}\Big\}, \{i, u\}\Big]\Big]\Bigg) +$$

$$\sum_{k=0}^{\infty}\sum_{j=0}^{k}\text{Piecewise}\Bigg[\Bigg\{\Bigg\{\frac{\text{Zeta}[n-k+j]}{(k-j)\,!}\Bigg(\text{KroneckerDelta}[j] + \frac{\text{UnitStep}[j-1]}{j\,!}$$

$$\text{BellY}\Big[\text{Table}\Big[\Big\{(-1)^{i}\,\text{Pochhammer}[j-k, i], \frac{i\,!\;(-1)^{i}}{i+1}\Big\},$$

$$\{i, j\}\Big]\Big]\Bigg), j \neq 1+k-n\Bigg\}\Bigg\}, 0\Bigg]\;(z-1)^{k}$$

$$\text{PolyLog}[n, z] == (-1)^{n-1}\sum_{k=1}^{\infty}\frac{1}{k^{n}\,z^{k}} - \frac{\text{Log}[-z]^{n}}{n\,!} +$$

$$2\sum_{k=1}^{\text{Floor}\left[\frac{n}{2}\right]}\frac{\text{PolyLog}[2k, -1]\;\text{Log}[-z]^{n-2k}}{(n-2k)\,!}\;\;/;\;\text{Abs}[z] > 1 \land n \in \text{Integers} \land n > 0$$

$$\text{PolyLog}[n, z] == \frac{\text{Log}[z]^{n-1}}{(n-1)\,!}\Big(\text{PolyGamma}[n] + \text{EulerGamma} - \text{Log}\Big[\text{Log}\Big[\frac{1}{z}\Big]\Big]\Big) +$$

$$\sum_{k=0}^{n-2}\frac{\text{Zeta}[n-k]}{k\,!}\,\text{Log}[z]^{k} + \sum_{k=n}^{\infty}\frac{\text{Zeta}[n-k]}{k\,!}\,\text{Log}[z]^{k}\;\;/;\;n \in \text{Integers} \land n > 0$$

$$\text{PolyLog}[n, z] == (-1)^{n-1} \sum_{k=1}^{\infty} \frac{1}{k^n z^k} - \frac{(2 \pi i)^n}{n!} \text{BernoulliB}\left[n, \frac{\text{Log}[-z]}{2 \pi i} + \frac{1}{2}\right] /;$$

$$\text{Abs}[z] > 1 \land n \in \text{Integers} \land n > 0$$

$$\text{PolyLog}[n, z] == \frac{\text{EulerGamma} + \text{PolyGamma}[n] - \text{Log}[-\text{Log}[z]]}{(n-1)!} \text{Log}[z]^{n-1} +$$

$$\text{Zeta}[n] + \sum_{j=1}^{n-2} \frac{\text{Zeta}[n-j]}{j!} \text{Log}[z]^j + \sum_{j=n}^{\infty} \frac{\text{Zeta}[n-j]}{j!} \text{Log}[z]^j \ /; n \in \text{Integers} \land n > 0$$

$$\text{PolyLog}[n, z] == \sum_{j=1}^{\infty} \text{Residue}\left[\frac{\text{Gamma}[s+1] \ (-z)^{-s}}{(-s)^n} \text{Gamma}[-s], \{s, j\}\right] /;$$

$$\text{Abs}[z] > 1 \land n \in \text{Integers} \land n > 0$$

Evaluation of FoxH in logarithmic cases (Residues of FoxH’s ratios of gamma functions)

Case of left u-th order poles

$$\text{Residue}\left[\frac{\left(\prod_{j=1}^{m} \text{Gamma}[b_j + \beta_j s]\right) \prod_{i=1}^{n} \text{Gamma}[1 - a_i - \alpha_i s]}{\left(\prod_{i=n+1}^{p} \text{Gamma}[a_i + \alpha_i s]\right) \prod_{j=m+1}^{q} \text{Gamma}[1 - b_j - \beta_j s]} z^{-s}, \left\{s, -\frac{b_m + \grave{\imath}_m}{\beta_m}\right\}\right] == \frac{z^{\frac{b_m + \grave{\imath}_m}{\beta_m}} \ (-1)^{\sum_{j=1}^{u} \grave{\imath}_{m-j+1}} \pi^u}{(u-1)!}$$

$$\left(\sum_{k=0}^{u-1} \text{Binomial}[u-1, k] \left(\frac{\text{KroneckerDelta}[u-1-k]}{\left(\prod_{i=n+1}^{p} \text{Gamma}\left[a_i - \frac{\alpha_i}{\beta_m} (\grave{\imath}_m + b_m)\right]\right) \prod_{j=m-u+1}^{q} \text{Gamma}\left[1 - b_j + \frac{\beta_j}{\beta_m} (\grave{\imath}_m + b_m)\right]} + \right.\right.$$

$$\text{UnitStep}[u-k-2] \ \text{BellY}\Big[\text{Table}\Big[$$

$$\left\{(-1)^j j! \left(\left(\prod_{i=n+1}^{p} \text{Gamma}\left[a_i - \frac{\alpha_i}{\beta_m} (\grave{\imath}_m + b_m)\right]\right) \prod_{j=m-u+1}^{q} \text{Gamma}\left[1 - b_j + \frac{\beta_j}{\beta_m} (\grave{\imath}_m + b_m)\right]\right)^{-1-j},\right.$$

$$\sum_{i=0}^{j} \text{Binomial}[j, i] \left(\sum_{k_{m-u+1}=0}^{j-i} \ldots \sum_{k_q=0}^{j-i} \text{KroneckerDelta}\left[j - i, \sum_{j=m-u+1}^{q} k_j\right]\right.$$

$$\text{Multinomial}[k_{m-u+1}, \ldots, k_q] \prod_{j=m-u+1}^{q} \left(\text{Gamma}\left[1 - b_j + \frac{\beta_j}{\beta_m} (\grave{\imath}_m + b_m)\right]\right.$$

$$(-\beta_j)^{k_j} \left(\text{KroneckerDelta}[k_j] + \text{UnitStep}[k_j - 1] \ \text{BellY}\Big[\text{Table}\Big[\Big\{1,\right.$$

$$\left.\left.\left.\text{PolyGamma}\left[-1 + t, 1 - b_j + \frac{\beta_j}{\beta_m} (\grave{\imath}_m + b_m)\right]\right\}, \{t, k_j\}\Big]\Big]\right)\right)\right)$$

$$\left(\sum_{k_{n+1}=0}^{i} \ldots \sum_{k_p=0}^{i} \text{KroneckerDelta}\left[i, \sum_{j=n+1}^{p} k_j\right] \text{Multinomial}[k_{n+1}, \ldots, k_p] \prod_{j=n+1}^{p} \left(\text{Gamma}\left[a_j - \frac{\alpha_j}{\beta_m} (\grave{\imath}_m + b_m)\right] \alpha_j^{k_j} \left(\text{KroneckerDelta}[k_j] + \text{UnitStep}[k_j - 1] \text{BellY}\left[\text{Table}\left[\left\{1, \text{PolyGamma}\left[-1 + t, a_j - \frac{\alpha_j}{\beta_m} (\grave{\imath}_m + b_m)\right]\right\}, \{t, k_j\}\right]\right]\right)\right)\right\}, \{j, u - 1 - k\}\right]\right)$$

$$\sum_{i=0}^{k}\sum_{j=0}^{k}\left(\text{Multinomial}[i, j, k - i - j] \text{Piecewise}\left[\{\{1, m === u\}\}, \left(\sum_{k_1=0}^{i} \ldots \sum_{k_{m-u}=0}^{i} \text{KroneckerDelta}\left[i, \sum_{j=1}^{m-u} k_j\right] \text{Multinomial}[k_1, \ldots, k_{m-u}] \prod_{j=1}^{m-u} \left(\text{Gamma}\left[b_j - \frac{\beta_j}{\beta_m} (\grave{\imath}_m + b_m)\right] \beta_j^{k_j} \left(\text{KroneckerDelta}[k_j] + \text{UnitStep}[k_j - 1] \text{BellY}\left[\text{Table}\left[\left\{1, \text{PolyGamma}\left[t - 1, b_j - \frac{\beta_j}{\beta_m} (\grave{\imath}_m + b_m)\right]\right\}, \{t, k_j\}\right]\right]\right)\right)\right)\right]$$

$$\left(\sum_{k_1=0}^{j} \ldots \sum_{k_n=0}^{j} \text{KroneckerDelta}\left[j, \sum_{j=1}^{n} k_j\right] \text{Multinomial}[k_1, \ldots, k_n] \prod_{j=1}^{n}\left(\text{Gamma}\left[1 - a_j + \frac{\alpha_j}{\beta_m} (\grave{\imath}_m + b_m)\right] (-\alpha_j)^{k_j} \left(\text{KroneckerDelta}[k_j] + \text{UnitStep}[k_j - 1] \text{BellY}\left[\text{Table}\left[\left\{1, \text{PolyGamma}\left[t - 1, 1 - a_j + \frac{\alpha_j}{\beta_m} (\grave{\imath}_m + b_m)\right]\right\}, \{t, k_j\}\right]\right]\right)\right)\right)$$

$$\left(\frac{\pi^{-u}}{\prod_{j=1}^{u} \beta_{m-j+1}} \sum_{r=0}^{\text{Floor}\left[\frac{k-i-j}{2}\right]} \frac{(k - i - j)!\ (-\text{Log}[z])^{k-i-j-2r}}{r!\ (k - i - j - 2r)!} \sum_{k_1=0}^{r} \ldots \sum_{k_u=0}^{r} \text{KroneckerDelta}\left[r, \sum_{j=1}^{u} k_j\right] \text{Multinomial}[k_1, \ldots, k_u] \prod_{j=1}^{u} \frac{2 (2^{2 k_j} - 2) \text{Zeta}[2 k_j] (\pi \beta_{m-j+1})^{2 k_j} k_j!}{(2\pi)^{2 k_j}}\right)\right)\right) /;$$

$u \in \text{Integers} \&\& \grave{\imath}_{m-j+1} \in \text{Integers} \&\& \grave{\imath}_{m-j+1} \geq$
$0 \&\&$
$1 \leq$
$j \leq$
$u \leq$
$m \&\&$
$b_{m-j+1} ==$

$\frac{(b_m + \grave{\imath}_m)\ \beta_{m-j+1}}{\beta_m}$ −

$\grave{\imath}_{m-j+1}$ &&

0 ≤

j ≤

u ≤

m &&

Not[

$a_i -$

$\frac{b_m + \grave{\imath}_m}{\beta_m}\ \alpha_i \in$

Integers && $a_i - \frac{b_m + \grave{\imath}_m}{\beta_m}\ \alpha_i \leq 0$ && n +

1 ≤ i ≤ p] &&

Not[$1 - b_j + \frac{b_m + \grave{\imath}_m}{\beta_m}\ \beta_j \in$ Integers && $1 - b_j + \frac{b_m + \grave{\imath}_m}{\beta_m}\ \beta_j \leq$

0 &&

m + 1 ≤ j ≤ q]

$$\operatorname{res}\left(\frac{\left(\prod_{j=1}^{m}\Gamma\left(b_j + s\,\beta_j\right)\right)\prod_{i=1}^{n}\Gamma(1 - a_i - \alpha_i\, s)}{\left(\prod_{i=n+1}^{p}\Gamma(a_i + s\,\alpha_i)\right)\prod_{j=m+1}^{q}\Gamma\left(1 - b_j - \beta_j\, s\right)}\ z^{-s},\ \left\{s, -\frac{b_m + \grave{\imath}_m}{\beta_m}\right\}\right) =$$

$$\frac{z^{\frac{b_m+\grave{\imath}_m}{\beta_m}}\ (-1)^{\sum_{j=1}^{u}\grave{\imath}_{m-j+1}}\ \pi^u}{(u-1)!}\sum_{k=0}^{u-1}\binom{u-1}{k}\left(\frac{\delta_{u-k-1}}{\left(\prod_{i=n+1}^{p}\Gamma\left(a_i - \frac{\alpha_i\,(b_m+\grave{\imath}_m)}{\beta_m}\right)\right)\prod_{j=m-u+1}^{q}\Gamma\left(1 - b_j + \frac{(b_m+\grave{\imath}_m)\,\beta_j}{\beta_m}\right)} + \right.$$

$$\theta(u-k-2)\ \mathrm{BellY}\Bigg[\mathrm{Table}\Bigg[\Bigg\{(-1)^j\ j!\left(\left(\prod_{i=n+1}^{p}\Gamma\left(a_i - \frac{\alpha_i\,(b_m+\grave{\imath}_m)}{\beta_m}\right)\right)\prod_{j=m-u+1}^{q}\Gamma\left(1 - b_j + \frac{(b_m+\grave{\imath}_m)\,\beta_j}{\beta_m}\right)\right)^{-j-1},$$

$$\sum_{i=0}^{j}\binom{j}{i}\left(\sum_{k_{m-u+1}=0}^{j-i}\cdots\sum_{k_q=0}^{j-i}\delta_{j-i,\sum_{j=m-u+1}^{q}k_j}\ (k_{m-u+1} + \ldots + k_q;\ k_{m-u+1},\ \ldots,\ k_q)\ \prod_{j=m-u+1}^{q}\Gamma\left(1 - b_j + \frac{(b_m+\grave{\imath}_m)\,\beta_j}{\beta_m}\right)\right.$$

$$\left.(-\beta_j)^{k_j}\left(\delta_{k_j} + \theta(k_j - 1)\ \mathrm{BellY}\left[\mathrm{Table}\left[\left\{1,\ \psi^{(t-1)}\left(1 - b_j + \frac{(b_m+\grave{\imath}_m)\,\beta_j}{\beta_m}\right)\right\},\ \{t,\ k_j\}\right]\right]\right)\right)$$

$$\sum_{k_{n+1}=0}^{i}\cdots\sum_{k_p=0}^{i}\delta_{i,\sum_{j=n+1}^{p}k_j}\ (k_{n+1} + \ldots + k_p;\ k_{n+1},\ \ldots,\ k_p)\ \prod_{j=n+1}^{p}\Gamma\left(a_j - \frac{\alpha_j\,(b_m+\grave{\imath}_m)}{\beta_m}\right)\alpha_j^{k_j}$$

$$\left.\left(\delta_{k_j} + \theta(k_j - 1)\ \mathrm{BellY}\left[\mathrm{Table}\left[\left\{1,\ \psi^{(t-1)}\left(a_j - \frac{\alpha_j\,(b_m+\grave{\imath}_m)}{\beta_m}\right)\right\},\ \{t,\ k_j\}\right]\right]\right)\Bigg\},\ \{j,\ u-k-1\}\Bigg]\Bigg]\right)$$

$$\sum_{i=0}^{k}\sum_{j=0}^{k}\frac{1}{\prod_{j=1}^{u}\beta_{m-j+1}}\,(k;\,i,\,j,\,k-i-j)\left\{\overset{1}{\begin{array}{l}\sum_{k_1=0}^{i}\ldots\sum_{k_{m-u}=0}^{i}\delta_{i,\sum_{j=1}^{m-u}k_j}\,(k_1+\ldots+k_{m-u};\,k_1,\,\ldots,\,k_{m-u})\\ \quad\prod_{j=1}^{m-u}\Gamma\left(b_j-\frac{\beta_j\,(b_m+\text{ì}_m)}{\beta_m}\right)\beta_j^{k_j}\left(\delta_{k_j}+\theta(k_j-1)\,\text{BellY}\left[\text{Table}\left[\left\{1,\,\psi^{(t-1)}\left(b_j-\frac{\beta_j\,(b_{\cdot}}{\beta}\right.\right.\right.\right.\right.\end{array}}\right.$$

$$\left.\frac{(b_m+\text{ì}_m)\,\alpha_j}{\beta_m}\right)(-\alpha_j)^{k_j}\left(\delta_{k_j}+\theta(k_j-1)\,\text{BellY}\left[\text{Table}\left[\left\{1,\,\psi^{(t-1)}\left(1-a_j+\frac{(b_m+\text{ì}_m)\,\alpha_j}{\beta_m}\right)\right\},\,\{t,\,k_j\}\right]\right]\right)\Bigg)$$

$$\left(\pi^{-u}\sum_{r=0}^{\left\lfloor\frac{k-i-j}{2}\right\rfloor}\frac{(k-i-j)!\,(-\log(z))^{k-i-j-2r}}{r!\,(k-i-j-2r)!}\sum_{k_1=0}^{r}\ldots\sum_{k_u=0}^{r}\delta_{r,\sum_{j=1}^{u}k_j}\,(k_1+\ldots+k_u;\,k_1,\,\ldots,\,k_u)\right.$$

$$\left.\prod_{j=1}^{u}\frac{2\,(2^{2k_j}-2)\,\zeta(2\,k_j)\,(\pi\,\beta_{-j+m+1})^{2k_j}\,k_j!}{(2\,\pi)^{2k_j}}\right)/;$$

$$u\in\mathbb{N}\land\text{ì}_{m-j+1}\in\mathbb{N}\land 1\leq j\leq u\leq m\land b_{m-j+1}=\frac{(b_m+\text{ì}_m)\,\beta_{m-j+1}}{\beta_m}-\text{ì}_{m-j+1}\land 0\leq$$

$$j\leq$$
$$u\leq$$
$$m\land$$

$$\neg\left(a_i-\frac{\alpha_i\,(b_m+\text{ì}_m)}{\beta_m}\in\mathbb{Z}\land\right.$$
$$a_i-\frac{\alpha_i\,(b_m+\text{ì}_m)}{\beta_m}\leq 0\land$$
$$\left.n+1\leq i\leq p\right)\land$$

$$\neg\left(\frac{\beta_j\,(b_m+\text{ì}_m)}{\beta_m}-b_j+1\in\mathbb{Z}\land\frac{\beta_j\,(b_m+\text{ì}_m)}{\beta_m}-b_j+1\leq 0\land m+1\leq j\leq q\right)$$

```
Quiet[
 Simplify[Table[With[{m = 6, n = 1, p = 2, q = 7},
             {Solve[Table[Subscript[b, m - j + 1] + Subscript[β, m - j + 1] * s ==
                        -Subscript[ì, m - j + 1] + ϵ * Subscript[β, m - j + 1], {j, u}],
    Union[{s}, Table[Subscript[b, m - j + 1], {j, 2, u}]]][[1]],
                  (Residue1[(Product[Gamma[Subscript[b, j] + Subscript[β, j] * s], {j, 1, m}] *
     Product[Gamma[1 - Subscript[a, i] - Subscript[α, i] * s], {i, 1, n}]) /
                         (Product[Gamma[Subscript[a, i] + Subscript[α, i] * s], {i, n + 1, p}] *
     Product[Gamma[1 - Subscript[b, j] - Subscript[β, j] * s], {j, m + 1, q}]) / z^s,
                        {ϵ, 0}, Assumptions → {And @@
                           Flatten[Union[Table[{Element[Subscript[ì, m - j + 1], Integers]},
      {j, 1, u}], Table[{Subscript[ì, m - j + 1] ≥ 0}, {j, 1, u}]]]}] /.
                       {s, -((Subscript[b, m] + Subscript[ì, m]) / Subscript[β, m])} → {ϵ, 0}) /
   (((z^((Subscript[b, m] + Subscript[ì, m]) / Subscript[β, m]) *
                           (-1)^Sum[Subscript[ì, m - j + 1], {j, 1, u}] * Pi^u) / (u - 1)!) *
```

```
Sum[Binomial[u - 1, k] * (KroneckerDelta[u - 1 - k] / (Product[
        Gamma[Subscript[a, i] - (Subscript[α, i] / Subscript[β, m]) *
          (Subscript[ì, m] + Subscript[b, m])], {i, n + 1, p}] *
  Product[Gamma[1 - Subscript[b, j] + (Subscript[β, j] / Subscript[β, m]) *
          (Subscript[ì, m] + Subscript[b, m])],
        {j, m - u + 1, q}]) + UnitStep[u - k - 2] *
  BellY[Table[{(-1)^j * j! * (Product[Gamma[Subscript[a, i] -
            (Subscript[α, i] / Subscript[β, m]) * (Subscript[
            ì, m] + Subscript[b, m])], {i, n + 1, p}] *
   Product[Gamma[1 - Subscript[b, j] + (Subscript[β, j] / Subscript[β, m]) * (Subscript[ì, m] +
            Subscript[b, m])], {j, m - u + 1, q}])^(-1 - j),
  Sum[Binomial[j, i] * (Sum[KroneckerDelta[j - i, Sum[Subscript[k, j], {j, m - u + 1, q}]] *
            Multinomial @@ Table[
            Subscript[k, j], {j, m - u + 1, q}] *
   Product[Gamma[1 - Subscript[b, j] + (Subscript[β, j] / Subscript[β, m]) *
            (Subscript[ì, m] + Subscript[b, m])] *
            (-Subscript[β, j])^Subscript[k, j] *
    (KroneckerDelta[Subscript[k, j]] + BellY[Table[{1, PolyGamma[-1 + t, 1 - Subscript[b, j] +
            (Subscript[β, j] / Subscript[β, m]) *
            (Subscript[ì, m] + Subscript[b, m])]},
     {t, Subscript[k, j]}]] * UnitStep[-1 + Subscript[k, j]]), {j, m - u + 1, q}], ##1] &) @@
          Table[{Subscript[k, j], 0, j - i}, {j, m - u + 1, q}] *
   (Sum[KroneckerDelta[i, Sum[Subscript[k, j], {j, n + 1, p}]] * Multinomial @@
            Table[Subscript[k, j], {j, n + 1, p}] * Product[
            Gamma[Subscript[a, j] - (Subscript[α, j] /
     Subscript[β, m]) * (Subscript[ì, m] + Subscript[b, m])] * Subscript[α, j]^Subscript[k, j] *
            (KroneckerDelta[Subscript[k, j]] +
            BellY[Table[{1, PolyGamma[-1 + t,
     Subscript[a, j] - (Subscript[α, j] / Subscript[β, m]) * (Subscript[ì, m] + Subscript[b,
            m])]}, {t, Subscript[k, j]}]] * UnitStep[
            -1 + Subscript[k, j]]), {j, n + 1, p}],
   ##1] &) @@ Table[{Subscript[k, j], 0, i}, {j, n + 1, p}], {i, 0, j}]}, {j, u - 1 - k}]]) * Sum[
        Multinomial[i, j, k - i - j] * Piecewise[{{1, m === u}},
  (Sum[KroneckerDelta[i, Sum[Subscript[k, j], {j, 1, m - u}]] * Multinomial @@
          Table[Subscript[k, j], {j, m - u}] * Product[Gamma[
            Subscript[b, j] - (Subscript[β, j] / Subscript[β, m]) *
    (Subscript[ì, m] + Subscript[b, m])] * Subscript[β, j]^Subscript[k, j] *
          (KroneckerDelta[Subscript[k, j]] + BellY[
            Table[{1, PolyGamma[-1 + r, Subscript[b, j] -
     (Subscript[β, j] / Subscript[β, m]) * (Subscript[ì, m] + Subscript[b, m])]},
            {r, Subscript[k, j]}]] * UnitStep[-1 +
```

```
                                              {1, Subscript[k, j]}]] * UnitStep[-1 +
                                              Subscript[k, j]]), {j, 1, m - u}], ##1] &) @@
        Table[{Subscript[k, j], 0, i}, {j, m - u}]] * (Sum[KroneckerDelta[j, Sum[Subscript[k, j], {j, 1,
                                          n}]] * Multinomial @@ Table[Subscript[k, j], {j, n}] *
         Product[Gamma[1 - Subscript[a, j] + (Subscript[α, j] / Subscript[β, m]) * (Subscript[ì, m] +
                                              Subscript[b, m])] * (-Subscript[α, j]) ^Subscript[
                                           k, j] * (KroneckerDelta[Subscript[k, j]] +
          BellY[Table[{1, PolyGamma[-1 + t, 1 - Subscript[a, j] + (Subscript[α, j] / Subscript[β, m]) *
                                              (Subscript[ì, m] + Subscript[b, m])]},
                                              {t, Subscript[k, j]}]] *
          UnitStep[-1 + Subscript[k, j]]), {j, 1, n}], ##1] &) @@
                                  Table[{Subscript[k, j], 0, j}, {j, n}] *
                                 ((1 / (Pi^u * Product[Subscript[β, m - j + 1], {j, 1, u}])) *
       Sum1[(((k - i - j)! * (-Log[z]) ^(k - i - j - 2 * r))/(k - i - j - 2 * r)!) *
                                     (1 / r!) * Sum1[Join[{KroneckerDelta[r,
                                           Sum[Subscript[k, j], {j, 1, u}]] * Multinomial[
          Sequence @@ Table[Subscript[k, i], {i, u}]] * Product[((2 * (2^(2 * Subscript[k, j]) - 2) *
                                              Zeta[2 * Subscript[k, j]] * (Pi * Subscript[
                                              β, m - j + 1]) ^(2 * Subscript[k, j])) /
           (2 * Pi) ^(2 * Subscript[k, j])) * Subscript[k, j]!, {j, u}]},
                                         Table[{Subscript[k, i], 0, r}, {i, u}]]],
                                     {r, 0, Floor[(k - i - j)/2]}]), {i, 0, k}, {j, 0, k}],
      {k, 0, u - 1}])} /. Sum1[uu_List] :> Sum1[Sequence @@ uu] /. Sum1 → Sum /.
    Solve[Table[Subscript[b, m - j + 1] + Subscript[β, m - j + 1] * s ==
                         -Subscript[ì, m - j + 1] + ϵ * Subscript[β, m - j + 1], {j, u}],
                    Union[{s}, Table[Subscript[b, m - j + 1], {j, 2, u}]]][[
    1]]], {u, 1, 2}] /. Residue1 → Residue] /. Gamma[1 + (w_)] :> w!]

{{{-((Subscript[b, 6] + Subscript[ì, 6] - ϵ * Subscript[β, 6]) / Subscript[β, 6]) →
      -((Subscript[b, 6] + Subscript[ì, 6] - ϵ * Subscript[β, 6]) / Subscript[β, 6])}, 1},
 {{-((Subscript[b, 6] + Subscript[ì, 6] - ϵ * Subscript[β, 6]) / Subscript[β, 6]) →
      -((Subscript[b, 6] + Subscript[ì, 6] - ϵ * Subscript[β, 6]) / Subscript[β, 6]),
   (Subscript[b, 6] * Subscript[β, 5] + Subscript[ì, 6] * Subscript[β, 5] -
          Subscript[ì, 5] * Subscript[β, 6]) / Subscript[β, 6] →
    (Subscript[b, 6] * Subscript[β, 5] + Subscript[ì, 6] * Subscript[β, 5] -
          Subscript[ì, 5] * Subscript[β, 6]) / Subscript[β, 6]}, 1}}
```

```
Clear[restrictedMultidimensionalSum]; restrictedMultidimensionalSum[body_, k_, {Q_, M_}] :=
 Module[{body1, kIterators},
   body1 = body /. Subscript[k, Q] → M - Sum[Subscript[k, j], {j, 1, Q - 1}];
  kIterators = Table[{Subscript[k, j], 0, M - Sum[Subscript[k, i], {i, 1, j - 1}]}, {j, Q - 1}];
  If[Q === 1, body /. Subscript[k, j_] :→ M,
     Sum[Evaluate[body1], Evaluate[Sequence @@ kIterators]]]]
Quiet[Simplify[
   Table[{Residue[(Product[Gamma[Subscript[b, j] - (Subscript[β, j] / Subscript[β, m]) * (Subscript[
                          ì, m] + Subscript[b, m]) + Subscript[β, j] * ϵ], {j, 1, m - u}] *
    Product[Gamma[1 - Subscript[a, i] + (Subscript[α, i] / Subscript[β, m]) *
                       (Subscript[ì, m] + Subscript[b, m]) - Subscript[α, i] * ϵ], {i, 1, n}] *
    Product[Csc[ϵ * Pi * Subscript[β, m - j + 1]], {j, 1, u}]) /
              (z^ϵ * (Product[Gamma[Subscript[a, i] - (Subscript[α, i] / Subscript[β, m]) *
                          (Subscript[ì, m] + Subscript[b, m]) + Subscript[α, i] * ϵ],
     {i, n + 1, p}] * Product[Gamma[1 - Subscript[b, j] +
                        (Subscript[β, j] / Subscript[β, m]) * (Subscript[ì, m] + Subscript[b, m]) -
                        Subscript[β, j] * ϵ], {j, m - u + 1, q}])), {ϵ, 0}] /
  Module[{pp, qq, res0, res}, pp[k_, u_, r_] :=
              restrictedMultidimensionalSum[Multinomial[Sequence @@ Table[Subscript[k, i], {i, u}]] *
     Product[(((2^(2 * Subscript[k, j]) - 2) * Zeta[2 * Subscript[k, j]] *
                          (Pi * Subscript[β, m - j + 1])^(2 * Subscript[k, j])) /
                       (2 * Pi)^(2 * Subscript[k, j])) * Subscript[k, j]!, {j, u}], k, {u, r}];
   qq[k_, u_] := ((k! * 2^u) / (Pi^u * Product[Subscript[β, m - j + 1], {j, 1, u}])) *
                Sum[((-Log[z])^(k - 2 * r) / (r! * (k - 2 * r)!)) * pp[k, u, r], {r, 0, Floor[k / 2]}];
   res0 = (1 / (u - 1)!) * Sum[Binomial[u - 1, k] * (KroneckerDelta[u - 1 - k] /
                          (Product[Gamma[Subscript[a, i] - (Subscript[α, i] / Subscript[β, m]) *
                                   (Subscript[ì, m] + Subscript[b, m])],
        {i, n + 1, p}] * Product[Gamma[1 - Subscript[b, j] + (Subscript[β, j] / Subscript[β, m]) *
                                   (Subscript[ì, m] + Subscript[b, m])],
                             {j, m - u + 1, q}]) + UnitStep[u - k - 2] *
       BellY[Table[{(-1)^j * j! * (Product[Gamma[Subscript[a, i] -
                                             (Subscript[α, i] / Subscript[β, m]) * (Subscript[ì,
                                              m] + Subscript[b, m])], {i, n + 1, p}] * Product[
          Gamma[1 - Subscript[b, j] + (Subscript[β, j] / Subscript[β, m]) *
                                              (Subscript[ì, m] + Subscript[b, m])],
                                        {j, m - u + 1, q}])^(-1 - j), Sum[Binomial[j, i] *
         restrictedMultidimensionalSum[Multinomial @@ Table[Subscript[k, j], {j, p - n}] *
                                    Product[Gamma[Subscript[a, j] - (Subscript[α, j] /
                                              Subscript[β, m]) * (Subscript[ì, m] +
           Subscript[b, m])] * Subscript[α, j]^Subscript[k, j - n] * (KroneckerDelta[Subscript[k,
                                              i - n]] + BellY[Table1[{1, PolyGamma[
```

```
j – n] ] + BellY[Table1[{1, PolyGamma[
                    –1 + t, Subscript[a, j] – (Subscript[α, j] /
    Subscript[β, m]) * (Subscript[ì, m] + Subscript[b, m])]}, {t, Subscript[k, j – n]}]]),
                  {j, n + 1, p}], k,
                {p – n, i}] * restrictedMultidimensionalSum[
  Multinomial @@ Table[Subscript[k, j], {j, q – m + u}] * Product[Gamma[1 – Subscript[b, j] +
                    (Subscript[β, j] / Subscript[β, m]) *
                    (Subscript[ì, m] + Subscript[b, m])] *
   (–Subscript[β, j])^Subscript[k, j – m + u] * (KroneckerDelta[Subscript[k, j – m + u]] +
                    BellY[Table1[{1, PolyGamma[–1 + t,
                    1 – Subscript[b, j] + (Subscript[β, j] /
    Subscript[β, m]) * (Subscript[ì, m] + Subscript[b, m])]},
                    {t, Subscript[k, j – m + u]}]]), {j, m – u + 1,
                  q}], k, {q – m + u, j – i}], {i, 0, j}]} /.
  Table1 → Table /. BellY[{}] → 0, {j, u – k – 1}]]) *
            Sum[Multinomial[i, j, k – i – j] * restrictedMultidimensionalSum[
              Multinomial @@ Table[Subscript[k, j], {j, m – u}] *
  Product[Gamma[Subscript[b, j] – (Subscript[β, j] / Subscript[β, m]) *
                (Subscript[ì, m] + Subscript[b, m])] * Subscript[β, j]^
              Subscript[k, j] * (KroneckerDelta[Subscript[k, j]] +
  UnitStep[Subscript[k, j] – 1] * BellY[Table1[{1, PolyGamma[t – 1, Subscript[b, j] – (Subscript[
                    β, j] / Subscript[β, m]) * (Subscript[ì, m] +
                    Subscript[b, m])]}, {t, Subscript[k, j]}]]),
  {j, 1, m – u}], k, {m – u, i}] * restrictedMultidimensionalSum[
              Multinomial @@ Table[Subscript[k, j], {j, n}] * Product[
                Gamma[1 – Subscript[a, j] + (Subscript[α, j] / Subscript[β, m]) *
   (Subscript[ì, m] + Subscript[b, m])] * (–Subscript[α, j])^Subscript[k, j] *
               (KroneckerDelta[Subscript[k, j]] + UnitStep[Subscript[k, j] – 1] *
  BellY[Table1[{1, PolyGamma[t – 1, 1 – Subscript[a, j] + (Subscript[α, j] / Subscript[β, m]) *
                   (Subscript[ì, m] + Subscript[b, m])]},
                {t, Subscript[k, j]}]]), {j, 1, n}], k, {n, j}] *
 Derivative[k – i – j][f3][0], {i, 0, k}, {j, 0, k}], {k, 0, u – 1}] /. Table1 → Table /.
          BellY[{}] → 1;
res = res0 /. Derivative[s_.][f3][0] :→ qq[s, u] /. f3[0] :→
          Limit[(ϵ^u * Product[Csc[ϵ * Pi * Subscript[β, m – j + 1]], {j, 1, u}]) / z^ϵ, ϵ → 0];
      res]}, {q, 6, 7}, {m, 4, 5}, {p, 4, 5},
{n, 1, 3}, {u, 1, 3}]]]
```

```
Clear[restrictedMultidimensionalSum]; restrictedMultidimensionalSum[body_, k_, {Q_, M_}] :=
 Module[{body1, kIterators},
   body1 = body /. Subscript[k, Q] → M - Sum[Subscript[k, j], {j, 1, Q - 1}];
  kIterators = Table[{Subscript[k, j], 0, M - Sum[Subscript[k, i], {i, 1, j - 1}]}, {j, Q - 1}];
  If[Q === 1, body /. Subscript[k, j_] :→ M,
     Sum[Evaluate[body1], Evaluate[Sequence @@ kIterators]]]]
Ans = Quiet[Simplify[Table[With[{m = 6, n = 1, p = 2, q = 7},
          {Solve[Table[Subscript[b, m - j + 1] + Subscript[β, m - j + 1] * s ==
                        -Subscript[ì, m - j + 1] + ϵ * Subscript[β, m - j + 1], {j, u}],
     Union[{s}, Table[Subscript[b, m - j + 1], {j, 2, u}]]][[1]],
    Simplify[Residue1[(Product[Gamma[Subscript[b, j] + Subscript[β, j] * s], {j, 1, m}] *
                                  Product[Gamma[1 - Subscript[a, i] - Subscript[α, i] * s], {i, 1, n}]) /
      (Product[Gamma[Subscript[a, i] + Subscript[α, i] * s], {i, n + 1, p}] * Product[Gamma[1 -
                                         Subscript[b, j] - Subscript[β, j] * s], {j, m + 1, q}]) / z^s, {ϵ, 0},
     Assumptions → {And @@ Flatten[Union[Table[{Element[Subscript[ì, m - j + 1], Integers]}, {j, 1,
                                         u}], Table[{Subscript[ì, m - j + 1] ≥ 0}, {j, 1, u}]]]}]] /.
    Solve[Table[Subscript[b, m - j + 1] + Subscript[β, m - j + 1] * s ==
                          -Subscript[ì, m - j + 1] + ϵ * Subscript[β, m - j + 1], {j, u}],
                     Union[{s}, Table[Subscript[b, m - j + 1], {j, 2, u}]]][[1]],
    z^((Subscript[b, m] + Subscript[ì, m]) / Subscript[β, m]) *
                   (-1)^Sum[Subscript[ì, m - j + 1], {j, 1, u}] * Pi^u * Module[{pp, qq, res0, res},
     pp[k_, u_, r_] := restrictedMultidimensionalSum[
                          Multinomial[Sequence @@ Table[Subscript[k, i], {i, u}]] *
                            Product[(((2^(2 * Subscript[k, j]) - 2) * Zeta[2 * Subscript[k, j]] *
          (Pi * Subscript[β, m - j + 1])^(2 * Subscript[k, j])) / (2 * Pi)^(2 * Subscript[k, j])) *
                                Subscript[k, j]!, {j, u}], k, {u, r}];
     qq[k_, u_] := ((k! * 2^u) / (Pi^u * Product[Subscript[β, m - j + 1], {j, 1, u}])) *
                          Sum[((-Log[z])^(k - 2 * r) / (r! * (k - 2 * r)!)) * pp[k, u, r],
                            {r, 0, Floor[k / 2]}];
     res0 = (1 / (u - 1)!) * Sum[Binomial[u - 1, ḳ] * (KroneckerDelta[u - 1 - ḳ] /
                                         (Product[Gamma[Subscript[a, i] - (Subscript[α, i] / Subscript[β,
                                                         m]) * (Subscript[ì, m] + Subscript[b, m])],
          {i, n + 1, p}] * Product[Gamma[1 - Subscript[b, j] + (Subscript[β, j] / Subscript[β, m]) *
                                                     (Subscript[ì, m] + Subscript[b, m])],
                                              {j, m - u + 1, q}]) + UnitStep[u - ḳ - 2] *
          BellY[Table[{(-1)^j̣ * j̣! * (Product[Gamma[Subscript[a, i] -
                                                                  (Subscript[α, i] / Subscript[β, m]) * (Subscript[ì,
                                                                  m] + Subscript[b, m])], {i, n + 1, p}] * Product[
              Gamma[1 - Subscript[b, j] + (Subscript[β, j] / Subscript[β, m]) * (Subscript[ì, m] +
                                                                  Subscript[b, m])], {j, m - u + 1,
                                                                  q}])^(-1 - j), Sum[Binomial[j, i] *
```

```
q j ] ) / ( -1 - j), Sum[Binomial[j, i] *
restrictedMultidimensionalSum[Multinomial @@ Table[Subscript[k, j], {j, p - n}] *
    Product[Gamma[Subscript[a, j] - (Subscript[α, j] /
    Subscript[β, m]) * (Subscript[ì, m] +
    Subscript[b, m])] * Subscript[α, j] ^ Subscript[k, j - n] * (KroneckerDelta[Subscript[k,
    j - n]] + BellY[Table1[{1, PolyGamma[
    -1 + t, Subscript[a, j] - (Subscript[α, j] /
    Subscript[β, m]) * (Subscript[ì, m] + Subscript[b, m])]}, {t, Subscript[k, j - n]}]]),
    {j, n + 1, p}], k, {p - n, i}] *
    restrictedMultidimensionalSum[
    Multinomial @@ Table[Subscript[k, j], {j, q - m + u}] * Product[Gamma[1 - Subscript[b,
    j] + (Subscript[β, j] / Subscript[β, m]) *
    (Subscript[ì, m] + Subscript[b, m])] *
    (-Subscript[β, j]) ^ Subscript[k, j - m + u] * (KroneckerDelta[Subscript[k, j - m + u]] +
    BellY[Table1[{1, PolyGamma[-1 + t,
    1 - Subscript[b, j] + (Subscript[β, j] /
    Subscript[β, m]) * (Subscript[ì, m] + Subscript[b, m])]}, {t, Subscript[k,
    j - m + u]}]]), {j, m - u + 1, q}],
    k, {q - m + u, j - i}], {i, 0, j}]} /.
  Table1 → Table /. BellY[{}] → 0, {j, u - k - 1}]]) * Sum[
    Multinomial[i, j, k - i - j] * restrictedMultidimensionalSum[
    Multinomial @@ Table[Subscript[k, j], {j, m - u}] *
  Product[Gamma[Subscript[b, j] - (Subscript[β, j] / Subscript[β, m]) *
    (Subscript[ì, m] + Subscript[b, m])] * Subscript[β, j] ^
    Subscript[k, j] * (KroneckerDelta[Subscript[k, j]] +
  UnitStep[Subscript[k, j] - 1] * BellY[Table1[{1, PolyGamma[t - 1, Subscript[b, j] -
    (Subscript[β, j] / Subscript[β, m]) *
    (Subscript[ì, m] + Subscript[b, m])]},
  {t, Subscript[k, j]}]]), {j, 1, m - u}], k, {m - u, i}] * restrictedMultidimensionalSum[
    Multinomial @@ Table[Subscript[k, j], {j, n}] *
  Product[Gamma[1 - Subscript[a, j] + (Subscript[α, j] / Subscript[β, m]) * (Subscript[ì, m] +
    Subscript[b, m])] * (-Subscript[α, j]) ^
    Subscript[k, j] * (KroneckerDelta[Subscript[k, j]] +
  UnitStep[Subscript[k, j] - 1] * BellY[Table1[{1, PolyGamma[t - 1, 1 - Subscript[a, j] +
    (Subscript[α, j] / Subscript[β, m]) *
    (Subscript[ì, m] + Subscript[b, m])]},
  {t, Subscript[k, j]}]]), {j, 1, n}], k, {n, j}] * Derivative[k - i - j][f3][0], {i, 0, k},
    {j, 0, k}], {k, 0, u - 1}] /. Table1 → Table /. BellY[{}] → 1;
res = res0 /. Derivative[s_.][f3][0] :→ qq[s, u] /. f3[0] :→ Limit[(ϵ ^ u * Product[
    Csc[ϵ * Pi * Subscript[β, m - j + 1]], {j, 1, u}]) / z ^ ϵ, ϵ → 0]; res] /.
Solve[Table[Subscript[b, m - j + 1] + Subscript[β, m - j + 1] * s ==
```

```
                    -Subscript[ì, m - j + 1] + ϵ * Subscript[β, m - j + 1], {j, u}],
                 Union[{s}, Table[Subscript[b, m - j + 1], {j, 2, u}]]][[
    1]]}], {u, 1, 5}]]]
TableForm[FullSimplify[Table[{Ans[[u, 1]], Ans[[u, 2]] / Ans[[u, 3]]}, {u, 1, 5}] /. Residue1 → Residue]]
TableForm[{{{s → ϵ - (Subscript[b, 6] + Subscript[ì, 6]) / Subscript[β, 6]}, 1},
   {{s → ϵ - (Subscript[b, 6] + Subscript[ì, 6]) / Subscript[β, 6],
  Subscript[b, 5] → -Subscript[ì, 5] +
        ((Subscript[b, 6] + Subscript[ì, 6]) * Subscript[β, 5]) / Subscript[β, 6]}, 1},
 {{s → ϵ - (Subscript[b, 6] + Subscript[ì, 6]) / Subscript[β, 6], Subscript[b, 4] →
      -Subscript[ì, 4] + ((Subscript[b, 6] + Subscript[ì, 6]) * Subscript[β, 4]) / Subscript[β, 6],
  Subscript[b, 5] → -Subscript[ì, 5] +
        ((Subscript[b, 6] + Subscript[ì, 6]) * Subscript[β, 5]) / Subscript[β, 6]}, 1},
 {{s → ϵ - (Subscript[b, 6] + Subscript[ì, 6]) / Subscript[β, 6], Subscript[b, 3] →
      -Subscript[ì, 3] + ((Subscript[b, 6] + Subscript[ì, 6]) * Subscript[β, 3]) / Subscript[β, 6],
  Subscript[b, 4] →
      -Subscript[ì, 4] + ((Subscript[b, 6] + Subscript[ì, 6]) * Subscript[β, 4]) / Subscript[β, 6],
  Subscript[b, 5] → -Subscript[ì, 5] +
        ((Subscript[b, 6] + Subscript[ì, 6]) * Subscript[β, 5]) / Subscript[β, 6]}, 1},
 {{s → ϵ - (Subscript[b, 6] + Subscript[ì, 6]) / Subscript[β, 6], Subscript[b, 2] →
      -Subscript[ì, 2] + ((Subscript[b, 6] + Subscript[ì, 6]) * Subscript[β, 2]) / Subscript[β, 6],
  Subscript[b, 3] →
      -Subscript[ì, 3] + ((Subscript[b, 6] + Subscript[ì, 6]) * Subscript[β, 3]) / Subscript[β, 6],
  Subscript[b, 4] →
      -Subscript[ì, 4] + ((Subscript[b, 6] + Subscript[ì, 6]) * Subscript[β, 4]) / Subscript[β, 6],
  Subscript[b, 5] → -Subscript[ì, 5] +
        ((Subscript[b, 6] + Subscript[ì, 6]) * Subscript[β, 5]) / Subscript[β, 6]}, 1}}]
```

Case of right u-th order poles

$$\text{Residue}\left[\frac{\left(\prod_{j=1}^{m}\text{Gamma}[b_j+\beta_j s]\right)\prod_{i=1}^{n}\text{Gamma}[1-a_i-\alpha_i s]}{\left(\prod_{i=n+1}^{p}\text{Gamma}[a_i+\alpha_i s]\right)\prod_{j=m+1}^{q}\text{Gamma}[1-b_j-\beta_j s]}\, z^{-s},\left\{s,\frac{1-a_n+\grave{\imath}_n}{\alpha_n}\right\}\right]==$$

$$\frac{z^{\frac{-1+a_n-\grave{\imath}_n}{\alpha_n}}\,(-1)^{u+\sum_{j=1}^{u}\grave{\imath}_{n-j+1}}\,\pi^u}{(u-1)!}\left(\sum_{k=0}^{u-1}\text{Binomial}[u-1,k]\right.$$

$$\left(\frac{\text{KroneckerDelta}[u-1-k]}{\left(\prod_{i=n-u+1}^{p}\text{Gamma}\left[a_i-\frac{-1+a_n-\grave{\imath}_n}{\alpha_n}\alpha_i\right]\right)\prod_{j=m+1}^{q}\text{Gamma}\left[1-b_j+\frac{-1+a_n-\grave{\imath}_n}{\alpha_n}\beta_j\right]}+\text{UnitStep}[u-k-2]\right.$$

$$\text{BellY}\left[\text{Table}\left[\left\{(-1)^j j!\left(\left(\prod_{i=n-u+1}^{p}\text{Gamma}\left[a_i-\frac{-1+a_n-\grave{\imath}_n}{\alpha_n}\alpha_i\right]\right)\prod_{j=m+1}^{q}\text{Gamma}\left[1-b_j+\frac{-1+a_n-\grave{\imath}_n}{\alpha_n}\right.\right.\right.\right.\right.$$

$$\beta_j \Big]\Big)^{-1-j}, \sum_{i=0}^{j} \text{Binomial}[j, i] \left(\sum_{k_{m+1}=0}^{j-i} \ldots \sum_{k_q=0}^{j-i} \text{KroneckerDelta}\Big[j - i, \sum_{j=m+1}^{q} k_j \Big] \right.$$

$$\text{Multinomial}[k_{m+1}, \ldots, k_q] \prod_{j=m+1}^{q} \left(\text{Gamma}\Big[1 - b_j + \frac{-1 + a_n - \grave{i}_n}{\alpha_n} \beta_j \Big] \right.$$

$$(-\beta_j)^{k_j} \left(\text{KroneckerDelta}[k_j] + \text{UnitStep}[k_j - 1] \,\text{BellY}\Big[\text{Table}\Big[\right.$$

$$\left. \left. \left. \Big\{ 1, \text{PolyGamma}\Big[t - 1, 1 - b_j + \frac{-1 + a_n - \grave{i}_n}{\alpha_n} \beta_j \Big] \Big\}, \{t, k_j\} \Big] \Big] \right) \right) \right)$$

$$\left(\sum_{k_{n-u+1}=0}^{i} \ldots \sum_{k_p=0}^{i} \text{KroneckerDelta}\Big[i, \sum_{j=n-u+1}^{p} k_j \Big] \text{Multinomial}[k_{n-u+1}, \ldots, k_p] \right.$$

$$\prod_{j=n-u+1}^{p} \left(\text{Gamma}\Big[a_j - \frac{-1 + a_n - \grave{i}_n}{\alpha_n} \alpha_j \Big] \alpha_j^{k_j} \right.$$

$$\left(\text{KroneckerDelta}[k_j] + \text{UnitStep}[k_j - 1] \,\text{BellY}\Big[\text{Table}\Big[\Big\{ 1, \text{PolyGamma}\Big[\right.$$

$$\left. \left. \left. t - 1, a_j - \frac{-1 + a_n - \grave{i}_n}{\alpha_n} \alpha_j \Big] \Big\}, \{t, k_j\} \Big] \Big] \right) \right) \right\}, \{j, u - k - 1\} \Big] \Big] \Big)$$

$$\sum_{i=0}^{k} \sum_{j=0}^{k} \left(\text{Multinomial}[i, j, k - i - j] \,\text{Piecewise}\Big[\{\{1, m === u\}\}, \left(\sum_{k_1=0}^{i} \ldots \sum_{k_m=0}^{i} \text{KroneckerDelta}\Big[\right. \right.$$

$$i, \sum_{j=1}^{m} k_j \Big] \text{Multinomial}[k_1, \ldots, k_m]$$

$$\prod_{j=1}^{m} \left(\text{Gamma}\Big[b_j - \frac{-1 + a_n - \grave{i}_n}{\alpha_n} \beta_j \Big] \beta_j^{k_j} \left(\text{KroneckerDelta}[k_j] + \text{UnitStep}[k_j - 1] \,\text{BellY}\Big[\right. \right.$$

$$\left. \left. \left. \text{Table}\Big[\Big\{ 1, \text{PolyGamma}\Big[t - 1, b_j - \frac{-1 + a_n - \grave{i}_n}{\alpha_n} \beta_j \Big] \Big\}, \{t, k_j\} \Big] \Big] \right) \right) \right) \Big]$$

$$\left(\sum_{k_1=0}^{j} \ldots \sum_{k_{n-u}=0}^{j} \text{KroneckerDelta}\Big[j, \sum_{j=1}^{n-u} k_j \Big] \text{Multinomial}[k_1, \ldots, k_{n-u}] \prod_{j=1}^{n-u} \text{Gamma}\Big[\right.$$

$$1 - a_j + \frac{-1 + a_n - \grave{i}_n}{\alpha_n} \alpha_j \Big] (-\alpha_j)^{k_j} \left(\text{KroneckerDelta}[k_j] + \text{UnitStep}[k_j - 1] \right.$$

$$\left. \left. \text{BellY}\Big[\text{Table}\Big[\Big\{ 1, \text{PolyGamma}\Big[t - 1, 1 - a_j + \frac{-1 + a_n - \grave{i}_n}{\alpha_n} \alpha_j \Big] \Big\}, \{t, k_j\} \Big] \Big] \right) \right)$$

$$\left(\frac{\pi^{-u}}{\prod_{j=1}^{u} \alpha_{n-j+1}} \sum_{r=0}^{\text{Floor}\left[\frac{k-i-j}{2}\right]} \frac{(k - i - j)!\, (-\text{Log}[z])^{k-i-j-2r}}{r!\, (k - i - j - 2r)!} \sum_{k_1=0}^{r} \ldots \sum_{k_u=0}^{r} \text{KroneckerDelta}\Big[r, \sum_{j=1}^{u} k_j \Big] \right.$$

$$\text{Multinomial}[k_1, \ldots, k_u] \prod_{j=1}^{u} \frac{2\left(2^{2 k_j} - 2\right) \text{Zeta}[2 k_j] \left(\pi \alpha_{n-j+1}\right)^{2 k_j} k_j!}{(2\pi)^{2 k_j}} \Bigg)\Bigg)\Bigg) \;/;$$

$u \in \text{Integers} \,\&\&\, ì_{n-j+1} \in \text{Integers} \,\&\&\, ì_{n-j+1} \geq$
$0 \,\&\&$
$1 \leq$
$j \leq$
$u \leq$
$n \,\&\&$
$a_{n-j+1} ==$
$\frac{(-1 + a_n - ì_n)\, \alpha_{n-j+1}}{\alpha_n} +$
$1 +$
$ì_{n-j+1} \,\&\&$
$0 \leq$
$j \leq$
$u \leq$
$n \,\&\&$
$\text{Not}\Big[$
$a_i +$
$\frac{1 - a_n + ì_n}{\alpha_n} \alpha_i \in$
$\text{Integers} \,\&\&\, a_i + \frac{1 - a_n + ì_n}{\alpha_n} \alpha_i \leq 0 \,\&\&$
$n + 1 \leq i \leq p\Big] \,\&\&\, \text{Not}\Big[$
$1 - b_j - \frac{1 - a_n + ì_n}{\alpha_n} \beta_j \in$
$\text{Integers} \,\&\&$
$1 - b_j - \frac{1 - a_n + ì_n}{\alpha_n} \beta_j \leq$
$0 \,\&\&$
$m + 1 \leq j \leq q\Big]$

$$\operatorname{res}\left(\frac{\left(\prod_{j=1}^{m} \Gamma(b_j + s\,\beta_j)\right) \prod_{i=1}^{n} \Gamma(1 - a_i - \alpha_i\, s)}{\left(\prod_{i=n+1}^{p} \Gamma(a_i + s\,\alpha_i)\right) \prod_{j=m+1}^{q} \Gamma(1 - b_j - \beta_j\, s)} z^{-s}, \left\{s, \frac{-a_n + ì_n + 1}{\alpha_n}\right\}\right) =$$

$$\frac{z^{\frac{a_n - ì_n - 1}{\alpha_n}} (-1)^{u + \sum_{j=1}^{u} ì_{n-j+1}} \pi^u}{(u-1)!} \sum_{k=0}^{u-1} \binom{u-1}{k} \left(\frac{\delta_{u-k-1}}{\left(\prod_{i=n-u+1}^{p} \Gamma\left(a_i - \frac{(a_n - ì_n - 1)\,\alpha_i}{\alpha_n}\right)\right) \prod_{j=m+1}^{q} \Gamma\left(1 - b_j + \frac{(a_n - ì_n - 1)\,\beta_j}{\alpha_n}\right)} \right. +$$

$$\theta(u-k-2)\,\text{BellY}\Big[\text{Table}\Big[\Big\{(-1)^{j}\,j!\,\Big(\Big(\prod_{i=n-u+1}^{p}\Gamma\Big(a_i-\frac{(a_n-\text{ì}_n-1)\,\alpha_i}{\alpha_n}\Big)\Big)\prod_{j=m+1}^{q}\Gamma\Big(1-b_j+\frac{(a_n-\text{ì}_n-1)\,\beta_j}{\alpha_n}\Big)\Big)^{-j-1},$$

$$\sum_{i=0}^{j}\binom{j}{i}\Big(\sum_{k_{m+1}=0}^{j-i}\ldots\sum_{k_q=0}^{j-i}\delta_{j-i,\sum_{j=m+1}^{q}k_j}\,(k_{m+1}+\ldots+k_q;\,k_{m+1},\,\ldots,\,k_q)\prod_{j=m+1}^{q}\Gamma\Big(1-b_j+\frac{(a_n-\text{ì}_n-1)\,\beta_j}{\alpha_n}\Big)(-\beta_j)^{k_j}$$

$$\Big(\delta_{k_j}+\theta(k_j-1)\,\text{BellY}\Big[\text{Table}\Big[\Big\{1,\,\psi^{(t-1)}\Big(1-b_j+\frac{(a_n-\text{ì}_n-1)\,\beta_j}{\alpha_n}\Big)\Big\},\,\{t,\,k_j\}\Big]\Big]\Big)\Big)$$

$$\sum_{k_{n-u+1}=0}^{i}\ldots\sum_{k_p=0}^{i}\delta_{i,\sum_{j=n-u+1}^{p}k_j}\,(k_{n-u+1}+\ldots+k_p;\,k_{n-u+1},\,\ldots,\,k_p)\prod_{j=n-u+1}^{p}\Gamma\Big(a_j-\frac{(a_n-\text{ì}_n-1)\,\alpha_j}{\alpha_n}\Big)\alpha_j^{k_j}$$

$$\Big(\delta_{k_j}+\theta(k_j-1)\,\text{BellY}\Big[\text{Table}\Big[\Big\{1,\,\psi^{(t-1)}\Big(a_j-\frac{(a_n-\text{ì}_n-1)\,\alpha_j}{\alpha_n}\Big)\Big\},\,\{t,\,k_j\}\Big]\Big]\Big)\Big\},\,\{j,\,u-k-1\}\Big]\Big]\Big)$$

$$\sum_{i=0}^{k}\sum_{j=0}^{k}\frac{(k;\,i,\,j,\,k-i-j)}{\prod_{j=1}^{u}\alpha_{n-j+1}}\sum_{k_1=0}^{i}\ldots\sum_{k_m=0}^{i}\delta_{i,\sum_{j=1}^{m}k_j}\,(k_1+\ldots+k_m;\,k_1,\,\ldots,\,k_m)\prod_{j=1}^{m}\Gamma\Big(b_j-\frac{(a_n-\text{ì}_n-1)\,\beta_j}{\alpha_n}\Big)\beta_j^{k_j}$$

$$\Big(\delta_{k_j}+\theta(k_j-1)\,\text{BellY}\Big[\text{Table}\Big[\Big\{1,\,\psi^{(t-1)}\Big(b_j-\frac{(a_n-\text{ì}_n-1)\,\beta_j}{\alpha_n}\Big)\Big\},\,\{t,\,k_j\}\Big]\Big]\Big)$$

$$\Big(\sum_{k_1=0}^{j}\ldots\sum_{k_{n-u}=0}^{j}\delta_{j,\sum_{j=1}^{n-u}k_j}\,(k_1+\ldots+k_{n-u};\,k_1,\,\ldots,\,k_{n-u})\prod_{j=1}^{n-u}\Gamma\Big(1-a_j+\frac{\alpha_j\,(a_n-\text{ì}_n-1)}{\alpha_n}\Big)(-\alpha_j)^{k_j}$$

$$\Big(\delta_{k_j}+\theta(k_j-1)\,\text{BellY}\Big[\text{Table}\Big[\Big\{1,\,\psi^{(t-1)}\Big(1-a_j+\frac{\alpha_j\,(a_n-\text{ì}_n-1)}{\alpha_n}\Big)\Big\},\,\{t,\,k_j\}\Big]\Big]\Big)\Big)$$

$$\Big(\pi^{-u}\sum_{r=0}^{\left\lfloor\frac{k-i-j}{2}\right\rfloor}\frac{(k-i-j)!\,(-\log(z))^{k-i-j-2r}}{r!\,(k-i-j-2r)!}\sum_{k_1=0}^{r}\ldots\sum_{k_u=0}^{r}\delta_{r,\sum_{j=1}^{u}k_j}\,(k_1+\ldots+k_u;\,k_1,\,\ldots,\,k_u)$$

$$\prod_{j=1}^{u}\frac{2\,(2^{2k_j}-2)\,\zeta(2k_j)\,(\pi\,\alpha_{n-j+1})^{2k_j}\,k_j!}{(2\pi)^{2k_j}}\Big)\,/;$$

$$u\in\mathbb{N}\land\text{ì}_{n-j+1}\in\mathbb{N}\land 1\le j\le u\le n\land a_{n-j+1}=\frac{(a_n-\text{ì}_n-1)\,\alpha_{n-j+1}}{\alpha_n}+$$

$$\text{ì}_{n-j+1}+$$
$$1\land$$
$$0\le$$
$$j\le$$
$$u\le$$
$$n\land$$
$$\neg\Big(\frac{\alpha_i\,(-a_n+\text{ì}_n+1)}{\alpha_n}+a_i\in$$
$$\mathbb{Z}\land\frac{\alpha_i\,(-a_n+\text{ì}_n+1)}{\alpha_n}+a_i\le$$

$$0 \wedge n+1 \leq i \leq p\Big) \wedge$$

$$\neg\left(-\frac{\beta_j\,(-a_n + \grave{\imath}_n + 1)}{\alpha_n} - b_j + 1 \in \mathbb{Z} \wedge -\frac{\beta_j\,(-a_n + \grave{\imath}_n + 1)}{\alpha_n} - b_j + 1 \leq 0 \wedge m+1 \leq j \leq q\right)$$

```
Quiet[
 Simplify[Table[With[{m = 1, n = 6, p = 7, q = 2},
      {Solve[Table[1 - Subscript[a, n - j + 1] - Subscript[α, n - j + 1] * s ==
                -Subscript[ì, n - j + 1] - ϵ * Subscript[α, n - j + 1], {j, u}],
      Union[{s}, Table[Subscript[a, n - j + 1], {j, 2, u}]]][[1]],
           (Residue1[(Product[Gamma[Subscript[b, j] + Subscript[β, j] * s], {j, 1, m}] *
        Product[Gamma[1 - Subscript[a, i] - Subscript[α, i] * s], {i, 1, n}]) /
                  (Product[Gamma[Subscript[a, i] + Subscript[α, i] * s], {i, n + 1, p}] *
        Product[Gamma[1 - Subscript[b, j] - Subscript[β, j] * s], {j, m + 1, q}]) / z^s,
                {ϵ, 0}, Assumptions → {And @@
                   Flatten[Union[Table[{Element[Subscript[ì, n - j + 1], Integers]},
         {j, 1, u}], Table[{Subscript[ì, n - j + 1] ≥ 0}, {j, 1, u}]]]}] /.
               {s, (1 - Subscript[a, n] + Subscript[ì, n]) / Subscript[α, n]} → {ϵ, 0}) /
      (((z^((-1 + Subscript[a, n] - Subscript[ì, n]) / Subscript[α, n]) * (-1)^
                 (u + Sum[Subscript[ì, n - j + 1], {j, 1, u}]) * Pi^u) / (u - 1)!) *
      Sum[Binomial[u - 1, ḳ] * (KroneckerDelta[u - 1 - ḳ] / (Product[Gamma[Subscript[a, i] -
                        ((-1 + Subscript[a, n] - Subscript[ì, n]) / Subscript[
                            α, n]) * Subscript[α, i]], {i, n - u + 1, p}] *
        Product[Gamma[1 - Subscript[b, j] + ((-1 + Subscript[a, n] - Subscript[ì, n]) /
                          Subscript[α, n]) * Subscript[β, j]],
                    {j, m + 1, q}]) + UnitStep[u - ḳ - 2] *
        BellY[Table[{(-1)^j̣ * j̣! * (Product[Gamma[Subscript[a, i] - ((-1 + Subscript[a, n] -
                            Subscript[ì, n]) / Subscript[α, n]) *
                            Subscript[α, i]], {i, n - u + 1, p}] *
          Product[Gamma[1 - Subscript[b, j] + ((-1 + Subscript[a, n] - Subscript[ì, n]) / Subscript[α,
                            n]) * Subscript[β, j]], {j, m + 1, q}])^(-1 - j̣),
         Sum[Binomial[j̣, ị] * (Sum[KroneckerDelta[j̣ - ị, Sum[Subscript[k, j], {j, m + 1, q}]] *
                            Multinomial @@ Table[Subscript[k, j],
                            {j, m + 1, q}] * Product[
            Gamma[1 - Subscript[b, j] + ((-1 + Subscript[a, n] - Subscript[ì, n]) / Subscript[α, n]) *
                            Subscript[β, j]] * (-Subscript[β, j])^Subscript[
                            k, j] * (KroneckerDelta[Subscript[k, j]] +
             BellY[Table[{1, PolyGamma[-1 + k, 1 - Subscript[b, j] + ((-1 + Subscript[a, n] -
                            Subscript[ì, n]) / Subscript[α, n]) * Subscript[
                            β, j]]}, {k, Subscript[k, j]}]] *
```

```
p, j]]], {k, Subscript[k, j]}]] *
UnitStep[-1 + Subscript[k, j]]), {j, m + 1, q}], ##1] &) @@
Table[{Subscript[k, j], 0, j - i}, {j, m + 1, q}] *
(Sum[KroneckerDelta[i, Sum[Subscript[k, j],
{j, n - u + 1, p}]] * Multinomial @@ Table[Subscript[k, j], {j, n - u + 1, p}] * Product[
Gamma[Subscript[a, j] - ((-1 + Subscript[a, n] -
Subscript[ì, n]) / Subscript[α, n]) *
Subscript[α, j]] * Subscript[α, j]^Subscript[k, j] * (KroneckerDelta[Subscript[k, j]] +
BellY[Table[{1, PolyGamma[-1 + k,
Subscript[a, j] - ((-1 + Subscript[a, n] -
Subscript[ì, n]) / Subscript[α, n]) * Subscript[α, j]]}, {k, Subscript[k, j]}]] *
UnitStep[-1 + Subscript[k, j]]),
{j, n - u + 1, p}], ##1] &) @@
Table[{Subscript[k, j], 0, i}, {j, n - u + 1, p}], {i, 0, j}]}, {j, u - 1 - k}]]) *
Sum[Multinomial[i, j, k - i - j] *
(Sum[KroneckerDelta[i, Sum[Subscript[k, j], {j, 1, m}]] * Multinomial @@
Table[Subscript[k, j], {j, m}] * Product[Gamma[Subscript[
b, j] - ((-1 + Subscript[a, n] - Subscript[ì, n]) /
Subscript[α, n]) * Subscript[β, j]] * Subscript[β, j]^Subscript[k, j] *
(KroneckerDelta[Subscript[k, j]] +
BellY[Table[{1, PolyGamma[-1 + k, Subscript[b, j] -
((-1 + Subscript[a, n] - Subscript[ì, n]) / Subscript[α, n]) * Subscript[β, j]]},
{k, Subscript[k, j]}]] * UnitStep[-1 +
Subscript[k, j]]), {j, 1, m}], ##1] &) @@
Table[{Subscript[k, j], 0, i}, {j, m}] *
(Sum[KroneckerDelta[j, Sum[Subscript[k, j], {j, 1, n - u}]] *
Multinomial @@ Table[Subscript[k, j], {j, n - u}] *
Product[Gamma[1 - Subscript[a, j] + ((-1 + Subscript[a, n] - Subscript[ì, n]) /
Subscript[α, n]) * Subscript[α, j]] *
(-Subscript[α, j])^Subscript[k, j] *
(KroneckerDelta[Subscript[k, j]] + BellY[Table[{1, PolyGamma[-1 + k, 1 - Subscript[a, j] +
((-1 + Subscript[a, n] - Subscript[ì, n]) /
Subscript[α, n]) * Subscript[α, j]]},
{k, Subscript[k, j]}]] * UnitStep[-1 + Subscript[k, j]]), {j, 1, n - u}], ##1] &) @@
Table[{Subscript[k, j], 0, j}, {j, n - u}] *
((1 / (Pi^u * Product[Subscript[α, n - j + 1], {j, 1, u}])) * Sum1[(((k - i - j)! * (-Log[z])^
(k - i - j - 2 * r)) / (k - i - j - 2 * r)!) * (1 / r!) *
Sum1[Join[{KroneckerDelta[r, Sum[Subscript[k, j], {j, 1, u}]] * Multinomial[
Sequence @@ Table[Subscript[k, i], {i, u}]] *
Product[((2 * (2^(2 * Subscript[k, j]) - 2) *
Zeta[2 * Subscript[k, j]] *
```

```
      (Pi * Subscript[α, n - j + 1]) ^ (2 * Subscript[k, j])) / (2 * Pi) ^ (2 * Subscript[k, j])) *
                                    Subscript[k, j] !, {j, u}]},
                                Table[{Subscript[k, i], 0, r}, {i, u}]]],
    {r, 0, Floor[(k - i - j)/2]}]), {i, 0, k}, {j, 0, k}], {k, 0, u - 1}])} /.
                Sum1[uu_List] :→ Sum1[Sequence @@ uu] /. Sum1 → Sum /.
  Solve[Table[1 - Subscript[a, n - j + 1] - Subscript[α, n - j + 1] * s ==
                  -Subscript[ì, n - j + 1] - ϵ * Subscript[α, n - j + 1], {j, u}],
                Union[{s}, Table[Subscript[a, n - j + 1], {j, 2, u}]]][[
  1]]], {u, 1, 2}] /. Residue1 → Residue] /. Gamma[1 + (w_)] :→ w!]
```

```
{{{(1 - Subscript[a, 6] + Subscript[ì, 6] + ϵ * Subscript[α, 6]) / Subscript[α, 6] →
     (1 - Subscript[a, 6] + Subscript[ì, 6] + ϵ * Subscript[α, 6]) / Subscript[α, 6]}, 1},
 {{(1 - Subscript[a, 6] + Subscript[ì, 6] + ϵ * Subscript[α, 6]) / Subscript[α, 6] →
     (1 - Subscript[a, 6] + Subscript[ì, 6] + ϵ * Subscript[α, 6]) / Subscript[α, 6],
   1 + Subscript[ì, 5] + ((-1 + Subscript[a, 6] - Subscript[ì, 6]) * Subscript[α, 5]) / Subscript[α, 6] →
     1 + Subscript[ì, 5] + ((-1 + Subscript[a, 6] - Subscript[ì, 6]) * Subscript[α, 5]) /
   Subscript[α, 6]}, 1}}
```

```
Clear[restrictedMultidimensionalSum]; restrictedMultidimensionalSum[body_, k_, {Q_, M_}] :=
 Module[{body1, kIterators},
   body1 = body /. Subscript[k, Q] → M - Sum[Subscript[k, j], {j, 1, Q - 1}];
  kIterators = Table[{Subscript[k, j], 0, M - Sum[Subscript[k, i], {i, 1, j - 1}]}, {j, Q - 1}];
  If[Q === 1, body /. Subscript[k, j_] :→ M,
     Sum[Evaluate[body1], Evaluate[Sequence @@ kIterators]]]]
```

```
Quiet[Simplify[
   Table[{Residue[(Product[Gamma[Subscript[b, j] - (-1 + Subscript[a, n] - Subscript[ì, n]) /
                        Subscript[α, n] + Subscript[β, j] * ϵ], {j, 1, m}] *
     Product[Gamma[1 - Subscript[a, i] + (-1 + Subscript[a, n] - Subscript[ì, n]) / Subscript[α, n] -
                       Subscript[α, i] * ϵ], {i, 1, n - u}] *
                Product[Csc[ϵ * Pi * Subscript[β, m - j + 1]], {j, 1, u}]) /
    (z^ϵ * (Product[Gamma[Subscript[a, j] - (-1 + Subscript[a, n] - Subscript[ì, n]) /
                            Subscript[α, n] + Subscript[α, j] * ϵ], {j, n - u + 1, p}] *
      Product[Gamma[1 - Subscript[b, j] + (-1 + Subscript[a, n] - Subscript[ì, n]) / Subscript[α, n] -
                          Subscript[β, j] * ϵ], {j, m + 1, q}])), {ϵ, 0}] -
   Module[{pp, qq, res0, res}, pp[k_, u_, r_] :=
              restrictedMultidimensionalSum[Multinomial[Sequence @@ Table[Subscript[k, i], {i, u}]] *
      Product[(((2^(2 * Subscript[k, j]) - 2) * Zeta[2 * Subscript[k, j]] *
                            (Pi * Subscript[β, m - j + 1]) ^ (2 * Subscript[k, j])) /
                         (2 * Pi) ^ (2 * Subscript[k, j])) * Subscript[k, j] !, {j, u}], k, {u, r}];
   qq[k_, u_] := ((k! * 2^u) / (Pi^u * Product[Subscript[β, m - j + 1], {j, 1, u}])) *
               Sum[((-Log[z]) ^ (k - 2 * r) / (r! * (k - 2 * r)!)) * pp[k, u, r], {r, 0, Floor[k / 2]}];
```

```
res0 = (1 / (u - 1) !) * Sum[Binomial[u - 1, k] * (KroneckerDelta[u - 1 - k] /
    (Product[Gamma[Subscript[a, j] - (-1 + Subscript[a, n] -
          Subscript[ì, n]) / Subscript[α, n]], {j, n - u + 1, p}] *
    Product[Gamma[1 - Subscript[b, j] + (-1 + Subscript[a, n] - Subscript[ì, n]) /
          Subscript[α, n]], {j, m + 1, q}]) + UnitStep[u - k - 2] *
    BellY[Table[{(-1) ^j * j ! * (Product[Gamma[Subscript[a, j] - (-1 + Subscript[a, n] - Subscript[
              ì, n]) / Subscript[α, n]], {j, n - u + 1, p}] *
          Product[Gamma[1 - Subscript[b, j] +
      (-1 + Subscript[a, n] - Subscript[ì, n]) / Subscript[α, n]], {j, m + 1, q}]) ^(-1 - j),
        Sum[Binomial[j, i] * restrictedMultidimensionalSum[
      Multinomial @@ Table[Subscript[k, j], {j, p - n + u}] * Product[Gamma[Subscript[a, j] -
              (-1 + Subscript[a, n] - Subscript[ì, n]) /
              Subscript[α, n]] * Subscript[α, j] ^
        Subscript[k, j - n + u] * (KroneckerDelta[Subscript[k, j - n + u]] + BellY[
              Table1[{1, PolyGamma[-1 + t, Subscript[a, j] -
              (-1 + Subscript[a, n] - Subscript[ì, n]) /
          Subscript[α, n]]}, {t, Subscript[k, j - n + u]}]]]), {j, n - u + 1, p}], k, {p - n + u, i}] *
          restrictedMultidimensionalSum[
            Multinomial @@ Table[Subscript[k, j],
        {j, q - m}] * Product[Gamma[1 - Subscript[b, j] + (-1 + Subscript[a, n] -
              Subscript[ì, n]) / Subscript[α, n]] *
            (-Subscript[β, j]) ^Subscript[k, j - m] *
        (KroneckerDelta[Subscript[k, j - m]] + BellY[Table1[{1, PolyGamma[t - 1,
              1 - Subscript[b, j] + (-1 + Subscript[a, n] -
              Subscript[ì, n]) / Subscript[α, n]]},
         {t, Subscript[k, j - m]}]]]), {j, m + 1, q}], k, {q - m, j - i}], {i, 0, j}]} /.
          Table1 → Table /. BellY[{}] → 0, {j, u - k - 1}]]) *
    Sum[Multinomial[i, j, k - i - j] * restrictedMultidimensionalSum[
        Multinomial @@ Table[Subscript[k, j], {j, m}] *
          Product[Gamma[Subscript[b, j] - (-1 + Subscript[a, n] -
      Subscript[ì, n]) / Subscript[α, n]] * Subscript[β, j] ^Subscript[k, j] *
            (KroneckerDelta[Subscript[k, j]] + UnitStep[Subscript[k, j] - 1] *
      BellY[Table1[{1, PolyGamma[t - 1, Subscript[b, j] - (-1 + Subscript[a, n] - Subscript[ì, n]) /
              Subscript[α, n]]},
            {t, Subscript[k, j]}]]]), {j, 1, m}], k, {m, i}] *
    restrictedMultidimensionalSum[Multinomial @@ Table[Subscript[k, j], {j, n - u}] *
          Product[Gamma[1 - Subscript[a, j] +
            (-1 + Subscript[a, n] - Subscript[ì, n]) / Subscript[α, n]] *
      (-Subscript[α, j]) ^Subscript[k, j] *
            (KroneckerDelta[Subscript[k, j]] + UnitStep[Subscript[k, j] - 1] *
              BellY[Table1[{1, PolyGamma[t - 1, 1 - Subscript[a, j] +
```

```
(-1 + Subscript[a, n] - Subscript[ì, n]) / Subscript[α, n]]}, {t, Subscript[k, j]}]]), {j, 1,
                n - u}], k, {n - u, j}] * Derivative[k - i - j][f3][0], {i, 0, k}, {j,
    0, k}], {k, 0, u - 1}] /. Table1 → Table /. BellY[{}] → 1;
        res = res0 /. Derivative[s_.][f3][0] :→ qq3[s, u] /.
    f3[0] :→ Limit[(ϵ^u * Product[Csc[ϵ * Pi * Subscript[β, m - j + 1]], {j, 1, u}]) / z^ϵ, ϵ → 0];
        res]}, {q, 4, 5}, {m, 1, 3}, {p, 6, 7}, {n, 4, 5}, {u, 1, 3}]]]

Clear[restrictedMultidimensionalSum]; restrictedMultidimensionalSum[body_, k_, {Q_, M_}] :=
 Module[{body1, kIterators},
   body1 = body /. Subscript[k, Q] → M - Sum[Subscript[k, j], {j, 1, Q - 1}];
  kIterators = Table[{Subscript[k, j], 0, M - Sum[Subscript[k, i], {i, 1, j - 1}]}, {j, Q - 1}];
  If[Q === 1, body /. Subscript[k, j_] :→ M,
     Sum[Evaluate[body1], Evaluate[Sequence @@ kIterators]]]]

Ans = Quiet[Simplify[Table[With[{m = 1, n = 6, p = 7, q = 2},
      {Solve[Table[1 - Subscript[a, n - j + 1] - Subscript[α, n - j + 1] * s ==
              -Subscript[ì, n - j + 1] - ϵ * Subscript[α, n - j + 1], {j, u}],
     Union[{s}, Table[Subscript[a, n - j + 1], {j, 2, u}]]][[1]],
    Simplify[Residue1[(Product[Gamma[Subscript[b, j] + Subscript[β, j] * s], {j, 1, m}] *
                  Product[Gamma[1 - Subscript[a, i] - Subscript[α, i] * s], {i, 1, n}]) /
     (Product[Gamma[Subscript[a, i] + Subscript[α, i] * s], {i, n + 1, p}] * Product[Gamma[1 -
                      Subscript[b, j] - Subscript[β, j] * s], {j, m + 1, q}]) / z^s, {ϵ, 0},
     Assumptions → {And @@ Flatten[Union[Table[{Element[Subscript[ì, n - j + 1], Integers]}, {j, 1,
                      u}], Table[{Subscript[ì, n - j + 1] ≥ 0}, {j, 1, u}]]]}]] /.
    Solve[Table[1 - Subscript[a, n - j + 1] - Subscript[α, n - j + 1] * s ==
              -Subscript[ì, n - j + 1] - ϵ * Subscript[α, n - j + 1], {j, u}],
           Union[{s}, Table[Subscript[a, n - j + 1], {j, 2, u}]]][[
    1]], z^((-1 + Subscript[a, n] - Subscript[ì, n]) / Subscript[α, n]) *
           (-1)^(u + Sum[Subscript[ì, n - j + 1], {j, 1, u}]) * Pi^u *
    Module[{pp, qq, res0, res}, pp[k_, u_, r_] := restrictedMultidimensionalSum[
              Multinomial[Sequence @@ Table[Subscript[k, i], {i, u}]] *
      Product[(((2^(2 * Subscript[k, j]) - 2) * Zeta[2 * Subscript[k, j]] *
                    (Pi * Subscript[α, n - j + 1])^(2 * Subscript[k, j])) /
                 (2 * Pi)^(2 * Subscript[k, j])) * Subscript[k, j]!, {j, u}], k, {u, r}];
     qq[k_, u_] := ((k! * 2^u) / (Pi^u * Product[Subscript[α, n - j + 1], {j, 1, u}])) *
              Sum[((-Log[z])^(k - 2 * r) / (r! * (k - 2 * r)!)) * pp[k, u, r],
                {r, 0, Floor[k / 2]}];
     res0 = (1 / (u - 1)!) * Sum[Binomial[u - 1, k] * (KroneckerDelta[u - 1 - k] / (Product[Gamma[
                          Subscript[a, j] - ((-1 + Subscript[a, n] - Subscript[ì, n]) /
                              Subscript[α, n]) * Subscript[α, j]],
         {j, n - u + 1, p}] * Product[Gamma[1 - Subscript[b, j] +
                           ((-1 + Subscript[a, n] - Subscript[ì, n]) /
```

```
                                   Subscript[α, n]) * Subscript[β, j]], {j, m + 1, q}]) +
   UnitStep[u - k - 2] * BellY[Table[{(-1)^j * j! * (Product[Gamma[Subscript[a, j] - ((-1 +
                                       Subscript[a, n] - Subscript[ì, n]) / Subscript[α,
                                       n]) * Subscript[α, j]], {j, n - u + 1, p}] *
     Product[Gamma[1 - Subscript[b, j] + ((-1 + Subscript[a, n] - Subscript[ì, n]) /
                                       Subscript[α, n]) * Subscript[β, j]],
                                       {j, m + 1, q}])^(-1 - j),
    Sum[Binomial[j, i] * restrictedMultidimensionalSum[Multinomial @@ Table[Subscript[k, j],
                                       {j, p - n + u}] * Product[Gamma[
                                       Subscript[a, j] - ((-1 + Subscript[a, n] -
       Subscript[ì, n]) / Subscript[α, n]) * Subscript[α, j]] *
                                       Subscript[α, j]^Subscript[k, j - n + u] *
                                       (KroneckerDelta[Subscript[k, j - n + u]] + BellY[
      Table1[{1, PolyGamma[-1 + t, Subscript[a, j] - ((-1 + Subscript[a, n] - Subscript[ì,
                                       n]) / Subscript[α, n]) * Subscript[α, j]]},
                                       {t, Subscript[k, j - n + u]}]]),
      {j, n - u + 1, p}], k, {p - n + u, i}] * restrictedMultidimensionalSum[
                                   Multinomial @@ Table[Subscript[k, j], {j, q - m}] *
                                    Product[Gamma[1 - Subscript[b, j] +
      ((-1 + Subscript[a, n] - Subscript[ì, n]) / Subscript[α, n]) * Subscript[β, j]] *
                                       (-Subscript[β, j])^Subscript[k, j - m] *
                                       (KroneckerDelta[Subscript[k, j - m]] +
      BellY[Table1[{1, PolyGamma[t - 1, 1 - Subscript[b, j] + ((-1 + Subscript[a, n] -
                                       Subscript[ì, n]) / Subscript[α, n]) * Subscript[
                                       β, j]]}, {t, Subscript[k, j - m]}]]),
      {j, m + 1, q}], k, {q - m, j - i}], {i, 0, j}]} /. Table1 → Table /. BellY[{}] → 0,
                            {j, u - k - 1}]]) * Sum[Multinomial[i, j, k - i - j] *
   restrictedMultidimensionalSum[Multinomial @@ Table[Subscript[k, j], {j, m}] *
                           Product[Gamma[Subscript[b, j] - ((-1 + Subscript[a, n] -
                                       Subscript[ì, n]) / Subscript[α, n]) *
     Subscript[β, j]] * Subscript[β, j]^Subscript[k, j] * (KroneckerDelta[Subscript[k, j]] +
                              UnitStep[Subscript[k, j] - 1] *
                               BellY[Table1[{1, PolyGamma[t - 1, Subscript[b, j] -
      ((-1 + Subscript[a, n] - Subscript[ì, n]) / Subscript[α, n]) * Subscript[β, j]]},
                                   {t, Subscript[k, j]}]]),
                            {j, 1, m}], k, {m, i}] * restrictedMultidimensionalSum[
   Multinomial @@ Table[Subscript[k, j], {j, n - u}] * Product[
                            Gamma[1 - Subscript[a, j] + ((-1 + Subscript[a, n] - Subscript[
                                       ì, n]) / Subscript[α, n]) * Subscript[α, j]] *
    (-Subscript[α, j])^Subscript[k, j] * (KroneckerDelta[Subscript[k, j]] +
                              UnitStep[Subscript[k, j] - 1] *
```

```
UnitStep[Subscript[k, j] − 1] *
    BellY[Table1[{1, PolyGamma[t − 1, 1 − Subscript[a, j] +
      ((−1 + Subscript[a, n] − Subscript[ì, n]) / Subscript[α, n]) * Subscript[α, j]]},
        {t, Subscript[k, j]}]]),
    {j, 1, n − u}], k, {n − u, j}] * Derivative[k − i − j][f3][
  0], {i, 0, k}, {j, 0, k}], {k, 0, u − 1}] /. Table1 → Table /. BellY[{}] → 1;
      res = res0 /. Derivative[s_.][f3][0] ⧴ qq[s, u] /.
  f3[0] ⧴ Limit[(ϵ^u * Product[Csc[ϵ * Pi * Subscript[α, n − j + 1]], {j, 1, u}]) / z^ϵ, ϵ → 0];
      res] /.
  Solve[Table[1 − Subscript[a, n − j + 1] − Subscript[α, n − j + 1] * s ==
        −Subscript[ì, n − j + 1] − ϵ * Subscript[α, n − j + 1], {j, u}],
      Union[{s}, Table[Subscript[a, n − j + 1], {j, 2, u}]]][[
  1]]}], {u, 1, 5}]]]
Table[{Ans[[u, 1]], Ans[[u, 2]] / Ans[[u, 3]]}, {u, 1, 5}] /. Residue1 → Residue // FullSimplify // TableForm
TableForm[{{{s → ϵ + (1 − Subscript[a, 6] + Subscript[ì, 6]) / Subscript[α, 6]}, 1},
   {{s → ϵ + (1 − Subscript[a, 6] + Subscript[ì, 6]) / Subscript[α, 6],
  Subscript[a, 5] → 1 + Subscript[ì, 5] +
      ((−1 + Subscript[a, 6] − Subscript[ì, 6]) * Subscript[α, 5]) / Subscript[α, 6]}, 1},
 {{s → ϵ + (1 − Subscript[a, 6] + Subscript[ì, 6]) / Subscript[α, 6],
    Subscript[a, 4] → 1 + Subscript[ì, 4] +
      ((−1 + Subscript[a, 6] − Subscript[ì, 6]) * Subscript[α, 4]) / Subscript[α, 6],
  Subscript[a, 5] → 1 + Subscript[ì, 5] +
      ((−1 + Subscript[a, 6] − Subscript[ì, 6]) * Subscript[α, 5]) / Subscript[α, 6]}, 1},
 {{s → ϵ + (1 − Subscript[a, 6] + Subscript[ì, 6]) / Subscript[α, 6],
    Subscript[a, 3] → 1 + Subscript[ì, 3] +
      ((−1 + Subscript[a, 6] − Subscript[ì, 6]) * Subscript[α, 3]) / Subscript[α, 6],
  Subscript[a, 4] → 1 + Subscript[ì, 4] +
      ((−1 + Subscript[a, 6] − Subscript[ì, 6]) * Subscript[α, 4]) / Subscript[α, 6],
  Subscript[a, 5] → 1 + Subscript[ì, 5] +
      ((−1 + Subscript[a, 6] − Subscript[ì, 6]) * Subscript[α, 5]) / Subscript[α, 6]}, 1},
 {{s → ϵ + (1 − Subscript[a, 6] + Subscript[ì, 6]) / Subscript[α, 6],
    Subscript[a, 2] → 1 + Subscript[ì, 2] +
      ((−1 + Subscript[a, 6] − Subscript[ì, 6]) * Subscript[α, 2]) / Subscript[α, 6],
  Subscript[a, 3] → 1 + Subscript[ì, 3] +
      ((−1 + Subscript[a, 6] − Subscript[ì, 6]) * Subscript[α, 3]) / Subscript[α, 6],
  Subscript[a, 4] → 1 + Subscript[ì, 4] +
      ((−1 + Subscript[a, 6] − Subscript[ì, 6]) * Subscript[α, 4]) / Subscript[α, 6],
  Subscript[a, 5] → 1 + Subscript[ì, 5] +
      ((−1 + Subscript[a, 6] − Subscript[ì, 6]) * Subscript[α, 5]) / Subscript[α, 6]}, 1}}]
```

Residues of Meijer’s ratios of gamma functions

Case of left simple poles

$$\text{Residue}\left[\frac{\left(\prod_{k=1}^{m}\text{Gamma}[s+b_k]\right)\prod_{k=1}^{n}\text{Gamma}[1-s-a_k]}{\left(\prod_{k=n+1}^{p}\text{Gamma}[s+a_k]\right)\prod_{k=m+1}^{q}\text{Gamma}[1-s-b_k]}\,z^{-s},\{s,-b_1-u\}\right]==$$
$$\frac{(-1)^u\left(\prod_{k=2}^{m}\text{Gamma}[-u-b_1+b_k]\right)\prod_{k=1}^{n}\text{Gamma}[1+u-a_k+b_1]}{u!\left(\prod_{k=n+1}^{p}\text{Gamma}[-u+a_k-b_1]\right)\prod_{k=m+1}^{q}\text{Gamma}[1+u+b_1-b_k]}\,z^{u+b_1}\ /;$$
u ∈ Integers && u ≥ 0 && $-b_1+b_j$ ∉ Integers && 2 ≤ j ≤ m && ! $(a_j-b_1$ ∈ Integers && $-1+a_j-b_1\geq 0)$ &&
1 ≤ j ≤ n && ! $(u-a_j+b_1$ ∈ Integers && $u-a_j+b_1\geq 0)$ && n + 1 ≤ j ≤ p &&
! $(-u-b_1+b_j$ ∈ Integers && $-1-u-b_1+b_j\geq 0)$ && m + 1 ≤ j ≤ q

$$\text{res}\left(\frac{\left(\prod_{k=1}^{m}\Gamma(s+b_k)\right)\prod_{k=1}^{n}\Gamma(-s-a_k+1)}{\left(\prod_{k=n+1}^{p}\Gamma(s+a_k)\right)\prod_{k=m+1}^{q}\Gamma(-s-b_k+1)}\,z^{-s},\{s,-b_1-u\}\right)=\frac{(-1)^u\left(\prod_{k=2}^{m}\Gamma(-u-b_1+b_k)\right)\prod_{k=1}^{n}\Gamma(u-a_k+b_1+1)}{u!\left(\prod_{k=n+1}^{p}\Gamma(-u+a_k-b_1)\right)\prod_{k=m+1}^{q}\Gamma(u+b_1-b_k+1)}\,z^{b_1+u}\ /;$$
$$u\in\mathbb{N}\land b_j-b_1\notin\mathbb{Z}\land 2\leq j\leq m\land a_j-b_1-1\notin\mathbb{N}\land 1\leq j\leq n\land$$
$$-a_j+b_1+u\notin\mathbb{N}\land n+1\leq j\leq p\land b_j-b_1-u-1\notin\mathbb{N}\land m+1\leq j\leq q$$

$$\left(\text{Table}\left[\left\{\text{Residue}\left[\frac{\left(\prod_{k=1}^{n}\text{Gamma}[1-s-a_k]\right)\prod_{k=1}^{m}\text{Gamma}[s+b_k]}{\left(\prod_{k=n+1}^{p}\text{Gamma}[s+a_k]\right)\prod_{k=m+1}^{q}\text{Gamma}[1-s-b_k]}\,z^{-s},\{s,-b_1-u\},\text{Assumptions}\to\{u\in\text{Integers}\land u\geq 0\}\right]\Big/\right.\right.\right.$$
$$\left.\left.\left(\frac{(-1)^u\left(\prod_{k=1}^{n}\text{Gamma}[1+u-a_k+b_1]\right)\prod_{k=2}^{m}\text{Gamma}[-u-b_1+b_k]}{u!\left(\prod_{k=n+1}^{p}\text{Gamma}[-u+a_k-b_1]\right)\prod_{k=m+1}^{q}\text{Gamma}[1+u+b_1-b_k]}\,z^{u+b_1}\right)\right\},\{m,3\},\{n,0,2\},\{p,n,4\},\{q,m,4\}\right]//$$
$$\left.\text{Simplify}[\#,\text{Assumptions}\to\{k\in\text{Integers}\}]\ \&\right)//\ \text{Simplify}[\#,\text{Assumptions}\to\{u\in\text{Integers}\land u\geq 0\}]\ \&$$

$$\left(\text{Table}\left[\left\{\text{Residue}\left[\frac{\left(\prod_{k=1}^{n}\text{Gamma}[1-s-a_k]\right)\prod_{k=1}^{m}\text{Gamma}[s+b_k]}{\left(\prod_{k=n+1}^{p}\text{Gamma}[s+a_k]\right)\prod_{k=m+1}^{q}\text{Gamma}[1-s-b_k]}\,z^{-s},\{s,-b_1-u\},\text{Assumptions}\to\{u\in\text{Integers}\land u\geq 0\}\right]\Big/\right.\right.\right.$$
(GammaResidueLeft[{Table[b_i, {i, m}], Table[1 − a_i, {i, n}]},
{Table[a_i, {i, n + 1, p}], Table[1 − b_i, {i, m + 1, q}]}, {b_1, 1, u}, z])}, {m, 3},
{n, 0, 2}, {p, n, 4}, {q, m, 4}] // Simplify[#, Assumptions → {k ∈ Integers}] &) //
Simplify[#, Assumptions → {u ∈ Integers ∧ u ≥ 0}] &

$$\text{Residue}\left[\frac{\left(\prod_{k=1}^{n}\text{Gamma}[1-s-a_k]\right)\prod_{k=1}^{m}\text{Gamma}[s+b_k]}{\left(\prod_{k=n+1}^{p}\text{Gamma}[s+a_k]\right)\prod_{k=m+1}^{q}\text{Gamma}[1-s-b_k]}\,z^{-s},\{s,-b_1-u\}\right]==$$
GammaResidueLeft[{Table[b_i, {i, m}], Table[1 − a_i, {i, n}]},
{Table[a_i, {i, n + 1, p}], Table[1 − b_i, {i, m + 1, q}]}, {b_1, 1, u}, z] /;
u ∈ Integers && u ≥ 0 && $-b_1+b_j$ ∉ Integers && 2 ≤ j ≤ m && ! $(a_j-b_1$ ∈ Integers && $-1+a_j-b_1\geq 0)$ &&
1 ≤ j ≤ n && ! $(u-a_j+b_1$ ∈ Integers && $u-a_j+b_1\geq 0)$ && n + 1 ≤ j ≤ p &&
! $(-u-b_1+b_j$ ∈ Integers && $-1-u-b_1+b_j\geq 0)$ && m + 1 ≤ j ≤ q

$$\text{res}\left(\frac{\left(\prod_{k=1}^{m}\Gamma(s+b_k)\right)\prod_{k=1}^{n}\Gamma(-s-a_k+1)}{\left(\prod_{k=n+1}^{p}\Gamma(s+a_k)\right)\prod_{k=m+1}^{q}\Gamma(-s-b_k+1)}\,z^{-s},\{s,-b_1-u\}\right)=\text{GammaResidueLeft}(\{\text{Table}[b_i,\{i,m\}],\text{Table}[1-a_i,\{i,n\}]\},$$
$$\{\text{Table}[a_i,\{i,n+1,p\}],\text{Table}[1-b_i,\{i,m+1,q\}]\},\{b_1,1,u\},z)\ /;\ u\in\mathbb{N}\land b_j-b_1\notin\mathbb{Z}\land 2\leq j\leq m\land$$
$$a_j-b_1-1\notin\mathbb{N}\land 1\leq j\leq n\land -a_j+b_1+u\notin\mathbb{N}\land n+1\leq j\leq p\land b_j-b_1-u-1\notin\mathbb{N}\land m+1\leq j\leq q$$

$$\left(\mathrm{Table}\left[\left\{\mathrm{Residue}\left[\frac{\left(\prod_{k=1}^{n}\mathrm{Gamma}[1-s-a_k]\right)\prod_{k=1}^{m}\mathrm{Gamma}[s+b_k]}{\left(\prod_{k=n+1}^{p}\mathrm{Gamma}[s+a_k]\right)\prod_{k=m+1}^{q}\mathrm{Gamma}[1-s-b_k]}\,z^{-s},\ \{s,-b_1-u\},\ \mathrm{Assumptions}\to\{u\in\mathrm{Integers}\wedge u\geq 0\}\right]\Big/\left(\frac{(-1)^u\left(\prod_{k=1}^{n}\mathrm{Gamma}[1+u-a_k+b_1]\right)\prod_{k=2}^{m}\mathrm{Gamma}[-u-b_1+b_k]}{u!\left(\prod_{k=n+1}^{p}\mathrm{Gamma}[-u+a_k-b_1]\right)\prod_{k=m+1}^{q}\mathrm{Gamma}[1+u+b_1-b_k]}\,z^{u+b_1}\right)\right\},\ \{m,3\},\ \{n,0,2\},\ \{p,n,4\},\ \{q,m,4\}\right]\ //\ \mathrm{Simplify}[\#,\ \mathrm{Assumptions}\to\{k\in\mathrm{Integers}\}]\ \&\right)\ //\ \mathrm{Simplify}[\#,\ \mathrm{Assumptions}\to\{u\in\mathrm{Integers}\wedge u\geq 0\}]\ \&$$

$$\left(\mathrm{Table}\left[\left\{\mathrm{Residue}\left[\frac{\left(\prod_{k=1}^{n}\mathrm{Gamma}[1-s-a_k]\right)\prod_{k=1}^{m}\mathrm{Gamma}[s+b_k]}{\left(\prod_{k=n+1}^{p}\mathrm{Gamma}[s+a_k]\right)\prod_{k=m+1}^{q}\mathrm{Gamma}[1-s-b_k]}\,z^{-s},\ \{s,-b_1-u\},\ \mathrm{Assumptions}\to\{u\in\mathrm{Integers}\wedge u\geq 0\}\right]\Big/\right.\right.$$

(GammaResidueLeft[{Table[b_i, {i, m}], Table[1 − a_i, {i, n}] },
{Table[a_i, {i, n + 1, p}], Table[1 − b_i, {i, m + 1, q}] }, { b_1, 1, u}, z]) }, {m, 3},
{n, 0, 2}, {p, n, 4}, {q, m, 4}] // Simplify[#, Assumptions→ {k ∈ Integers}] &) //
Simplify[#, Assumptions→ {u ∈ Integers ∧ u ≥ 0}] &

Case of right simple poles

$$\mathrm{Residue}\left[\frac{\left(\prod_{k=1}^{m}\mathrm{Gamma}[s+b_k]\right)\prod_{k=1}^{n}\mathrm{Gamma}[1-s-a_k]}{\left(\prod_{k=n+1}^{p}\mathrm{Gamma}[s+a_k]\right)\prod_{k=m+1}^{q}\mathrm{Gamma}[1-s-b_k]}\,z^{-s},\ \{s,1-a_1+u\}\right] ==$$

$$\frac{(-1)^{-1+u}\left(\prod_{k=1}^{m}\mathrm{Gamma}[1+u-a_1+b_k]\right)\prod_{k=2}^{n}\mathrm{Gamma}[-u+a_1-a_k]}{u!\left(\prod_{k=n+1}^{p}\mathrm{Gamma}[1+u-a_1+a_k]\right)\prod_{k=m+1}^{q}\mathrm{Gamma}[-u+a_1-b_k]}\,z^{-1-u+a_1}\ /;$$

u ∈ Integers && u ≥ 0 && $a_1 - a_j$ ∉ Integers && 2 ≤ j ≤ n && ! ($a_1 - b_j$ ∈ Integers && $-1 + a_1 - b_j \geq 0$) &&
1 ≤ j ≤ m && ! ($u - a_1 + b_j$ ∈ Integers && $u - a_1 + b_j \geq 0$) && m + 1 ≤ j ≤ q &&
! ($-u + a_1 - a_j$ ∈ Integers && $-1 - u + a_1 - a_j \geq 0$) && n + 1 ≤ j ≤ p

$$\mathrm{res}_s\left(\frac{\left(\prod_{k=1}^{m}\Gamma(s+b_k)\right)\prod_{k=1}^{n}\Gamma(-s-a_k+1)}{\left(\prod_{k=n+1}^{p}\Gamma(s+a_k)\right)\prod_{k=m+1}^{q}\Gamma(-s-b_k+1)}\,z^{-s}\right)(1-a_1+u) == \frac{(-1)^{u-1}\left(\prod_{k=1}^{m}\Gamma(u-a_1+b_k+1)\right)\prod_{k=2}^{n}\Gamma(-u+a_1-a_k)}{u!\left(\prod_{k=n+1}^{p}\Gamma(u-a_1+a_k+1)\right)\prod_{k=m+1}^{q}\Gamma(-u+a_1-b_k)}\,z^{a_1-u-1}\ /;$$

$$u\in\mathbb{N}\wedge a_1-a_j\notin\mathbb{Z}\wedge 2\leq j\leq n\wedge a_1-b_j-1\notin\mathbb{N}\wedge 1\leq j\leq m\wedge -a_1+b_j+u\notin\mathbb{N}\wedge m+1\leq j\leq q\wedge -a_j+a_1-u-1\notin\mathbb{N}\wedge n+1\leq j\leq p$$

$$\left(\mathrm{Table}\left[\left\{\mathrm{Residue}\left[\frac{z^{-s}\left(\prod_{k=1}^{n}\mathrm{Gamma}[1-s-a_k]\right)\prod_{k=1}^{m}\mathrm{Gamma}[s+b_k]}{\left(\prod_{k=n+1}^{p}\mathrm{Gamma}[s+a_k]\right)\prod_{k=m+1}^{q}\mathrm{Gamma}[1-s-b_k]},\ \{s,1+u-a_1\},\ \mathrm{Assumptions}\to\{u\in\mathrm{Integers}\wedge u\geq 0\}\right]\Big/\left(\frac{(-1)^{-1+u}z^{-1-u+a_1}\left(\prod_{k=2}^{n}\mathrm{Gamma}[-u+a_1-a_k]\right)\prod_{k=1}^{m}\mathrm{Gamma}[1+u-a_1+b_k]}{u!\left(\prod_{k=n+1}^{p}\mathrm{Gamma}[1+u-a_1+a_k]\right)\prod_{k=m+1}^{q}\mathrm{Gamma}[-u+a_1-b_k]}\right)\right\},\ \{m,0,2\},\ \{n,2\},\ \{p,n,4\},\ \{q,m,4\}\right]\ //\ \mathrm{Simplify}[\#,\ \mathrm{Assumptions}\to\{k\in\mathrm{Integers}\}]\ \&\right)\ //\ \mathrm{Simplify}[\#,\ \mathrm{Assumptions}\to\{u\in\mathrm{Integers}\wedge u\geq 0\}]\ \&$$

$$\left(\mathrm{Table}\left[\left\{\mathrm{Residue}\left[\frac{z^{-s}\left(\prod_{k=1}^{n}\mathrm{Gamma}[1-s-a_k]\right)\prod_{k=1}^{m}\mathrm{Gamma}[s+b_k]}{\left(\prod_{k=n+1}^{p}\mathrm{Gamma}[s+a_k]\right)\prod_{k=m+1}^{q}\mathrm{Gamma}[1-s-b_k]},\ \{s,1+u-a_1\},\ \mathrm{Assumptions}\to\{u\in\mathrm{Integers}\wedge u\geq 0\}\right]\Big/\right.\right.$$

(GammaResidueRight[{Table[b_i, {i, m}], Table[1 − a_i, {i, n}] },
{Table[a_i, {i, n + 1, p}], Table[1 − b_i, {i, m + 1, q}] }, {1 − a_1, 1, u}, z]) },
{m, 0, 2}, {n, 2}, {p, n, 4}, {q, m, 4}] // Simplify[#, Assumptions→ {k ∈ Integers}] &) //
Simplify[#, Assumptions→ {u ∈ Integers ∧ u ≥ 0}] &

$$\mathrm{Residue}\left[\frac{\left(\prod_{k=1}^{m}\mathrm{Gamma}[s+b_k]\right)\prod_{k=1}^{n}\mathrm{Gamma}[1-s-a_k]}{\left(\prod_{k=n+1}^{p}\mathrm{Gamma}[s+a_k]\right)\prod_{k=m+1}^{q}\mathrm{Gamma}[1-s-b_k]}\,z^{-s},\ \{s,1-a_1+u\}\right] ==$$

```
GammaResidueRight[{Table[b_i, {i, m}], Table[1 - a_i, {i, n}]},
  {Table[a_i, {i, n + 1, p}], Table[1 - b_i, {i, m + 1, q}]}, {1 - a_1, 1, u}, z] /;
 u ∈ Integers && u ≥ 0 && a_1 - a_j ∉ Integers && 2 ≤ j ≤ n && ! (a_1 - b_j ∈ Integers && -1 + a_1 - b_j ≥ 0) &&
  1 ≤ j ≤ m && ! (u - a_1 + b_j ∈ Integers && u - a_1 + b_j ≥ 0) && m + 1 ≤ j ≤ q &&
  ! (-u + a_1 - a_j ∈ Integers && -1 - u + a_1 - a_j ≥ 0) && n + 1 ≤ j ≤ p
```

$$\operatorname{res}_s\left(\frac{\left(\prod_{k=1}^{m}\Gamma(s+b_k)\right)\prod_{k=1}^{n}\Gamma(-s-a_k+1)}{\left(\prod_{k=n+1}^{p}\Gamma(s+a_k)\right)\prod_{k=m+1}^{q}\Gamma(-s-b_k+1)}\,z^{-s}\right)(1-a_1+u) = \text{GammaResidueRight}(\{\text{Table}[b_i,\{i,m\}],\text{Table}[1-a_i,\{i,n\}]\}, \{\text{Table}[a_i,\{i,n+1,p\}],\text{Table}[1-b_i,\{i,m+1,q\}]\},\{1-a_1,1,u\},z)\ /;\ u\in\mathbb{N}\wedge a_1-a_j\notin\mathbb{Z}\wedge 2\le j\le n\wedge a_1-b_j-1\notin\mathbb{N}\wedge 1\le j\le m\wedge -a_1+b_j+u\notin\mathbb{N}\wedge m+1\le j\le q\wedge -a_j+a_1-u-1\notin\mathbb{N}\wedge n+1\le j\le p$$

```
(Table[{Residue[(z^-s (∏_{k=1}^n Gamma[1-s-a_k]) ∏_{k=1}^m Gamma[s+b_k]) /
      ((∏_{k=n+1}^p Gamma[s+a_k]) ∏_{k=m+1}^q Gamma[1-s-b_k]), {s, 1 + u - a_1}, Assumptions → {u ∈ Integers ∧ u ≥ 0}] /
     (((-1)^(-1+u) z^(-1-u+a_1) (∏_{k=2}^n Gamma[-u+a_1-a_k]) ∏_{k=1}^m Gamma[1+u-a_1+b_k]) /
      (u! (∏_{k=n+1}^p Gamma[1+u-a_1+a_k]) ∏_{k=m+1}^q Gamma[-u+a_1-b_k]))}, {m, 0, 2}, {n, 2}, {p, n, 4}, {q, m, 4}] //
   Simplify[#, Assumptions → {k ∈ Integers}] &) // Simplify[#, Assumptions → {u ∈ Integers ∧ u ≥ 0}] &
(Table[{Residue[(z^-s (∏_{k=1}^n Gamma[1-s-a_k]) ∏_{k=1}^m Gamma[s+b_k]) /
      ((∏_{k=n+1}^p Gamma[s+a_k]) ∏_{k=m+1}^q Gamma[1-s-b_k]), {s, 1 + u - a_1}, Assumptions → {u ∈ Integers ∧ u ≥ 0}] /
     (GammaResidueRight[{Table[b_i, {i, m}], Table[1 - a_i, {i, n}]},
       {Table[a_i, {i, n + 1, p}], Table[1 - b_i, {i, m + 1, q}]}, {1 - a_1, 1, u}, z])},
    {m, 0, 2}, {n, 2}, {p, n, 4}, {q, m, 4}] // Simplify[#, Assumptions → {k ∈ Integers}] &) //
 Simplify[#, Assumptions → {u ∈ Integers ∧ u ≥ 0}] &
```

Case of left double poles

```
Residue[((∏_{k=1}^m Gamma[s + b_k]) ∏_{k=1}^n Gamma[1 - s - a_k]) /
   ((∏_{k=n+1}^p Gamma[s + a_k]) ∏_{k=m+1}^q Gamma[1 - s - b_k]) z^-s, {s, -b_2 - u}] ==
 ((-1)^(-b_1+b_2) z^(u+b_2) (∏_{k=1}^n Gamma[1 + u - a_k + b_2]) (∏_{k=3}^m Gamma[-u - b_2 + b_k])) /
   (u! (u - b_1 + b_2)! (∏_{k=n+1}^p Gamma[-u + a_k - b_2]) ∏_{k=m+1}^q Gamma[1 + u + b_2 - b_k]) *
  (-Log[z] + PolyGamma[0, 1 + u] + PolyGamma[0, 1 + u - b_1 + b_2] -
    Sum_{k=n+1}^p PolyGamma[0, -u + a_k - b_2] - Sum_{k=1}^n PolyGamma[0, 1 + u - a_k + b_2] +
    Sum_{k=m+1}^q PolyGamma[0, 1 + u + b_2 - b_k] + Sum_{k=3}^m PolyGamma[0, -u - b_2 + b_k]) /;
 -b_1 + b_2 ∈ Integers && -b_1 + b_2 ≥ 0 && u ∈ Integers && u ≥ 0 && -b_1 + b_j ∉ Integers &&
  3 ≤ j ≤ m && ! (a_j - b_2 ∈ Integers && -1 + a_j - b_2 ≥ 0) && 1 ≤ j ≤ n &&
  ! (u - a_j + b_2 ∈ Integers && u - a_j + b_2 ≥ 0) && n + 1 ≤ j ≤ p &&
  ! (-u - b_2 + b_j ∈ Integers && -1 - u - b_2 + b_j ≥ 0) && m + 1 ≤ j ≤ q
```

$$\operatorname{res}\left(\frac{\left(\prod_{k=1}^{m}\Gamma(s+b_k)\right)\prod_{k=1}^{n}\Gamma(-s-a_k+1)}{\left(\prod_{k=n+1}^{p}\Gamma(s+a_k)\right)\prod_{k=m+1}^{q}\Gamma(-s-b_k+1)}\,z^{-s},\{s,-b_2-u\}\right) =$$

$$\frac{(-1)^{b_2-b_1}\, z^{b_2+u}\,(\prod_{k=3}^{m}\Gamma(-u-b_2+b_k))\prod_{k=1}^{n}\Gamma(u-a_k+b_2+1)}{u!\,(-b_1+b_2+u)!\left(\prod_{k=n+1}^{p}\Gamma(-u+a_k-b_2)\right)\prod_{k=m+1}^{q}\Gamma(u+b_2-b_k+1)}\left(-\sum_{k=n+1}^{p}\psi^{(0)}(-u+a_k-b_2)-\sum_{k=1}^{n}\psi^{(0)}(u-a_k+b_2+1)+\sum_{k=m+1}^{q}\psi^{(0)}(u+b_2-b_k+1)+\sum_{k=3}^{m}\psi^{(0)}(-u-b_2+b_k)+\psi^{(0)}(u-b_1+b_2+1)+\psi^{(0)}(u+1)-\log(z)\right)/;$$

$$b_2-b_1\in\mathbb{N}\wedge u\in\mathbb{N}\wedge b_j-b_1\notin\mathbb{Z}\wedge 3\le j\le m\wedge a_j-b_2-1\notin\mathbb{N}\wedge 1\le j\le n\wedge -a_j+b_2+u\notin\mathbb{N}\wedge n+1\le j\le p\wedge b_j-b_2-u-1\notin\mathbb{N}\wedge m+1\le j\le q$$

```
Assuming[b2 = b1 + h,
  (Table[{Residue[((∏_{k=1}^n Gamma[1-s-ak]) ∏_{k=1}^m Gamma[s+bk])/((∏_{k=n+1}^p Gamma[s+ak]) ∏_{k=m+1}^q Gamma[1-s-bk]) z^-s, {s, -b2 - u}, Assumptions→ {u ∈ Integers∧ u ≥ 0 ∧
            h ∈ Integers∧ h ≥ 0}] / (((-1)^(-b1+b2) z^(u+b2) (∏_{k=1}^n Gamma[1+u-ak+b2]) (∏_{k=3}^m Gamma[-u-b2+bk]))/(u! (u-b1+b2)! (∏_{k=n+1}^p Gamma[-u+ak-b2]) ∏_{k=m+1}^q Gamma[1+u+b2-bk]) *
          (-Log[z] + PolyGamma[0, 1 + u] + PolyGamma[0, 1 + u - b1 + b2] -
            ∑_{k=n+1}^p PolyGamma[0, -u + ak - b2] - ∑_{k=1}^n PolyGamma[0, 1 + u - ak + b2] +
            ∑_{k=m+1}^q PolyGamma[0, 1 + u + b2 - bk] + ∑_{k=3}^m PolyGamma[0, -u - b2 + bk]))}, {m, 2, 4},
       {n, 0, 2}, {p, n, 4}, {q, m, 4}] // Simplify[#, Assumptions→ {k ∈ Integers}] &) //
    Simplify[#, Assumptions→ {u ∈ Integers∧ u ≥ 0 ∧ h ∈ Integers∧ h ≥ 0}] &] //
  Simplify[
    #,
    Assumptions→
      {u ∈ Integers∧ u ≥ 0 ∧ h ∈ Integers∧ h ≥ 0}] &
Assuming[b2 = b1 + h,
  (Table[{Residue[((∏_{k=1}^n Gamma[1-s-ak]) ∏_{k=1}^m Gamma[s+bk])/((∏_{k=n+1}^p Gamma[s+ak]) ∏_{k=m+1}^q Gamma[1-s-bk]) z^-s, {s, -b2 - u}, Assumptions→ {u ∈ Integers∧ u ≥ 0 ∧
            h ∈ Integers∧ h ≥ 0}] / (GammaResidueLeft[{Table[bi, {i, m}], Table[1 - ai,
            {i, n}]}, {Table[ai, {i, n + 1, p}], Table[1 - bi, {i, m + 1, q}]}, {b2, 2, u}, z])},
       {m, 2, 4}, {n, 0, 2}, {p, n, 4}, {q, m, 4}] // Simplify[#, Assumptions→ {k ∈ Integers}] &) //
    Simplify[#, Assumptions→ {u ∈ Integers∧ u ≥ 0}] &] //
  Simplify[#, Assumptions→ {u ∈ Integers∧ u ≥ 0 ∧ h ∈ Integers∧ h ≥ 0}] &
Residue[((∏_{k=1}^m Gamma[s + bk]) ∏_{k=1}^n Gamma[1 - s - ak])/((∏_{k=n+1}^p Gamma[s + ak]) ∏_{k=m+1}^q Gamma[1 - s - bk]) z^-s, {s, -b2 - u}] ==
   GammaResidueLeft[{Table[bi, {i, m}], Table[1 - ai, {i, n}]},
     {Table[ai, {i, n + 1, p}], Table[1 - bi, {i, m + 1, q}]}, {b2, 2, u}, z] /;
  -b1 + b2 ∈ Integers&& -b1 + b2 ≥ 0 && u ∈ Integers&& u ≥ 0 && -b1 + bj ∉ Integers&& 3 ≤ j ≤ m &&
    ! (aj - b2 ∈ Integers&& -1 + aj - b2 ≥ 0) && 1 ≤ j ≤ n && ! (u - aj + b2 ∈ Integers&& u - aj + b2 ≥ 0) &&
    n + 1 ≤ j ≤ p && ! (-u - b2 + bj ∈ Integers&& -1 - u - b2 + bj ≥ 0) && m + 1 ≤ j ≤ q
```

$$\mathrm{res}\left(\frac{\left(\prod_{k=1}^{m}\Gamma(s+b_k)\right)\prod_{k=1}^{n}\Gamma(-s-a_k+1)}{\left(\prod_{k=n+1}^{p}\Gamma(s+a_k)\right)\prod_{k=m+1}^{q}\Gamma(-s-b_k+1)}\,z^{-s},\ \{s,-b_2-u\}\right)=\mathrm{GammaResidueLeft}($$
$$\{\mathrm{Table}[b_i,\{i,m\}],\ \mathrm{Table}[1-a_i,\{i,n\}]\},\ \{\mathrm{Table}[a_i,\{i,n+1,p\}],\ \mathrm{Table}[1-b_i,\{i,m+1,q\}]\},\ \{b_2,2,u\},\ z)\ /;$$
$$b_2-b_1\in\mathbb{N}\land u\in\mathbb{N}\land b_j-b_1\notin\mathbb{Z}\land 3\le j\le m\land a_j-b_2-1\notin\mathbb{N}\land 1\le j\le n\land -a_j+b_2+u\notin\mathbb{N}\land$$
$$n+1\le j\le p\land b_j-b_2-u-1\notin\mathbb{N}\land m+1\le j\le q$$

```
Assuming[b2 = b1 + h,
  (Table[{Residue[((∏_{k=1}^n Gamma[1-s-ak]) ∏_{k=1}^m Gamma[s+bk]) / ((∏_{k=n+1}^p Gamma[s+ak]) ∏_{k=m+1}^q Gamma[1-s-bk]) z^-s, {s, -b2 - u}, Assumptions→ {u ∈ Integers∧ u ≥ 0 ∧
          h ∈ Integers∧ h ≥ 0}] / (((-1)^(-b1+b2) z^(u+b2) (∏_{k=1}^n Gamma[1+u-ak+b2]) (∏_{k=3}^m Gamma[-u-b2+bk])) / (u! (u-b1+b2)! (∏_{k=n+1}^p Gamma[-u+ak-b2]) ∏_{k=m+1}^q Gamma[1+u+b2-bk]) *
        (-Log[z] + PolyGamma[0, 1 + u] + PolyGamma[0, 1 + u - b1 + b2] -
          ∑_{k=n+1}^p PolyGamma[0, -u + ak - b2] - ∑_{k=1}^n PolyGamma[0, 1 + u - ak + b2] +
          ∑_{k=m+1}^q PolyGamma[0, 1 + u + b2 - bk] + ∑_{k=3}^m PolyGamma[0, -u - b2 + bk]))}, {m, 2, 4},
      {n, 0, 2}, {p, n, 4}, {q, m, 4}] // Simplify[♯, Assumptions→ {k ∈ Integers}] &) //
    Simplify[♯, Assumptions→ {u ∈ Integers∧ u ≥ 0 ∧ h ∈ Integers∧ h ≥ 0}] &] //
 Simplify[
   ♯,
   Assumptions→
    {u ∈ Integers∧ u ≥ 0 ∧ h ∈ Integers∧ h ≥ 0}] &
Assuming[b2 = b1 + h,
  (Table[{Residue[((∏_{k=1}^n Gamma[1-s-ak]) ∏_{k=1}^m Gamma[s+bk]) / ((∏_{k=n+1}^p Gamma[s+ak]) ∏_{k=m+1}^q Gamma[1-s-bk]) z^-s, {s, -b2 - u}, Assumptions→ {u ∈ Integers∧ u ≥ 0 ∧
          h ∈ Integers∧ h ≥ 0}] / (GammaResidueLeft[{Table[bi, {i, m}], Table[1 - ai,
          {i, n}]}, {Table[ai, {i, n + 1, p}], Table[1 - bi, {i, m + 1, q}]}, {b2, 2, u}, z])},
      {m, 2, 4}, {n, 0, 2}, {p, n, 4}, {q, m, 4}] // Simplify[♯, Assumptions→ {k ∈ Integers}] &) //
    Simplify[♯, Assumptions→ {u ∈ Integers∧ u ≥ 0}] &] //
 Simplify[♯, Assumptions→ {u ∈ Integers∧ u ≥ 0 ∧ h ∈ Integers∧ h ≥ 0}] &
```

Case of right double poles

$$\mathrm{Residue}\left[\frac{\left(\prod_{k=1}^{m}\mathrm{Gamma}[s+b_k]\right)\prod_{k=1}^{n}\mathrm{Gamma}[1-s-a_k]}{\left(\prod_{k=n+1}^{p}\mathrm{Gamma}[s+a_k]\right)\prod_{k=m+1}^{q}\mathrm{Gamma}[1-s-b_k]}\,z^{-s},\ \{s,1-a_2+u\}\right]==$$
$$\frac{(-1)^{-1+a_1-a_2}\left(\prod_{k=1}^{m}\mathrm{Gamma}[1+u-a_2+b_k]\right)\prod_{k=3}^{n}\mathrm{Gamma}[-u+a_2-a_k]}{u!\ (u+a_1-a_2)!\ \left(\prod_{k=n+1}^{p}\mathrm{Gamma}[1+u-a_2+a_k]\right)\prod_{k=m+1}^{q}\mathrm{Gamma}[-u+a_2-b_k]}$$
$$z^{-1-u+a_2}\Big(\mathrm{Log}[z]+\mathrm{PolyGamma}[0,1+u]+\mathrm{PolyGamma}[0,1+u+a_1-a_2]+$$

$$\sum_{k=3}^{n} \text{PolyGamma}[0, -u + a_2 - a_k] + \sum_{k=n+1}^{p} \text{PolyGamma}[0, 1 + u - a_2 + a_k] - \sum_{k=m+1}^{q} \text{PolyGamma}[0, -u + a_2 - b_k] - \sum_{k=1}^{m} \text{PolyGamma}[0, 1 + u - a_2 + b_k]\Bigg) \;/;$$

$a_1 - a_2 \in \text{Integers} \&\& a_1 - a_2 \geq 0 \&\& u \in \text{Integers} \&\& u \geq 0 \&\& a_1 - a_j \notin \text{Integers} \&\& 3 \leq j \leq n \&\&$
$!\,(a_2 - b_j \in \text{Integers} \&\& -1 + a_2 - b_j \geq 0) \&\& 1 \leq j \leq m \&\& !\,(u - a_2 + b_j \in \text{Integers} \&\& u - a_2 + b_j \geq 0) \&\&$
$m + 1 \leq j \leq q \&\& !\,(-u + a_2 - a_j \in \text{Integers} \&\& -1 - u + a_2 - a_j \geq 0) \&\& n + 1 \leq j \leq p$

$$\text{res}_s\left(\frac{\left(\prod_{k=1}^{m} \Gamma(s+b_k)\right) \prod_{k=1}^{n} \Gamma(-s-a_k+1)}{\left(\prod_{k=n+1}^{p} \Gamma(s+a_k)\right) \prod_{k=m+1}^{q} \Gamma(-s-b_k+1)}\, z^{-s}\right)(1 - a_2 + u) ==$$

$$\frac{(-1)^{a_1-a_2-1} \left(\prod_{k=1}^{m} \Gamma(u-a_2+b_k+1)\right) \prod_{k=3}^{n} \Gamma(-u+a_2-a_k)}{u!\,(a_1-a_2+u)!\left(\prod_{k=n+1}^{p} \Gamma(u-a_2+a_k+1)\right) \prod_{k=m+1}^{q} \Gamma(-u+a_2-b_k)}\, z^{a_2-u-1}$$

$$\left(-\sum_{k=m+1}^{q} \psi^{(0)}(-u+a_2-b_k) - \sum_{k=1}^{m} \psi^{(0)}(u-a_2+b_k+1) + \sum_{k=n+1}^{p} \psi^{(0)}(u-a_2+a_k+1) + \sum_{k=3}^{n} \psi^{(0)}(-u+a_2-a_k) + \psi^{(0)}(u+a_1-a_2+1) + \psi^{(0)}(u+1) + \log(z)\right) /;\; a_1 - a_2 \in \mathbb{N} \wedge u \in \mathbb{N} \wedge a_1 - a_j \notin \mathbb{Z} \wedge 3 \leq j \leq n \wedge$$

$$a_2 - b_j - 1 \notin \mathbb{N} \wedge 1 \leq j \leq m \wedge -a_2 + b_j + u \notin \mathbb{N} \wedge m + 1 \leq j \leq q \wedge -a_j + a_2 - u - 1 \notin \mathbb{N} \wedge n + 1 \leq j \leq p$$

```
Assuming[a2 = a1 - h,
  (Table[{Residue[(z^-s (∏_{k=1}^m Gamma[s+bk]) ∏_{k=1}^n Gamma[1-s-ak]) / ((∏_{k=n+1}^p Gamma[s+ak]) ∏_{k=m+1}^q Gamma[1-s-bk]), {s, 1 - a2 + u}, Assumptions → {u ∈ Integers ∧ u ≥ 0 ∧
          h ∈ Integers ∧ h ≥ 0}] / (((-1)^(-1+a1-a2) (∏_{k=1}^m Gamma[1+u-a2+bk]) ∏_{k=3}^n Gamma[-u+a2-ak]) / (u! (u+a1-a2)! (∏_{k=n+1}^p Gamma[1+u-a2+ak]) ∏_{k=m+1}^q Gamma[-u+a2-bk])
        z^(-1-u+a2) (Log[z] + PolyGamma[0, 1 + u] + PolyGamma[0, 1 + u + a1 - a2] +
          ∑_{k=3}^n PolyGamma[0, -u + a2 - ak] + ∑_{k=n+1}^p PolyGamma[0, 1 + u - a2 + ak] -
          ∑_{k=m+1}^q PolyGamma[0, -u + a2 - bk] - ∑_{k=1}^m PolyGamma[0, 1 + u - a2 + bk]))}, {m, 0, 2},
    {n, 2, 3}, {p, n, 4}, {q, m, 4}] // Simplify[#, Assumptions → {k ∈ Integers}] &)] //
  Simplify[#, Assumptions → {u ∈ Integers ∧ u ≥ 0 ∧ h ∈ Integers ∧ h ≥ 0}] &
Assuming[a2 = a1 - h,
  (Table[{Residue[(z^-s (∏_{k=1}^m Gamma[s+bk]) ∏_{k=1}^n Gamma[1-s-ak]) / ((∏_{k=n+1}^p Gamma[s+ak]) ∏_{k=m+1}^q Gamma[1-s-bk]), {s, 1 - a2 + u}, Assumptions → {u ∈ Integers ∧ u ≥ 0 ∧
          h ∈ Integers ∧ h ≥ 0}] / (GammaResidueRight[{Table[bi, {i, m}], Table[1 - ai, {i, n}]},
        {Table[ai, {i, n + 1, p}], Table[1 - bi, {i, m + 1, q}]}, {1 - a2, 2, u}, z])}, {m, 0, 2},
    {n, 2, 3}, {p, n, 4}, {q, m, 4}] // Simplify[#, Assumptions → {k ∈ Integers}] &)] //
  Simplify[#, Assumptions → {u ∈ Integers ∧ u ≥ 0 ∧ h ∈ Integers ∧ h ≥ 0}] &
Residue[((∏_{k=1}^m Gamma[s + bk]) ∏_{k=1}^n Gamma[1 - s - ak]) / ((∏_{k=n+1}^p Gamma[s + ak]) ∏_{k=m+1}^q Gamma[1 - s - bk]) z^-s, {s, 1 - a2 + u}] ==
  GammaResidueRight[{Table[bi, {i, m}], Table[1 - ai, {i, n}]},
```

```
{Table[a_i, {i, n + 1, p}], Table[1 - b_i, {i, m + 1, q}]}, {1 - a_2, 2, u}, z] /;
a_1 - a_2 ∈ Integers && a_1 - a_2 ≥ 0 && u ∈ Integers && u ≥ 0 && a_1 - a_j ∉ Integers && 3 ≤ j ≤ n &&
! (a_2 - b_j ∈ Integers && -1 + a_2 - b_j ≥ 0) && 1 ≤ j ≤ m && ! (u - a_2 + b_j ∈ Integers && u - a_2 + b_j ≥ 0) &&
m + 1 ≤ j ≤ q && ! (-u + a_2 - a_j ∈ Integers && -1 - u + a_2 - a_j ≥ 0) && n + 1 ≤ j ≤ p
```

$$\operatorname{res}_s\left(\frac{\left(\prod_{k=1}^{m}\Gamma(s+b_k)\right)\prod_{k=1}^{n}\Gamma(-s-a_k+1)}{\left(\prod_{k=n+1}^{p}\Gamma(s+a_k)\right)\prod_{k=m+1}^{q}\Gamma(-s-b_k+1)}\,z^{-s}\right)(1-a_2+u) = \text{GammaResidueRight}(\{\text{Table}[b_i,\{i,m\}],\text{Table}[1-a_i,\{i,n\}]\},\{\text{Table}[a_i,\{i,n+1,p\}],\text{Table}[1-b_i,\{i,m+1,q\}]\},\{1-a_2,2,u\},z)\ /;$$
$$a_1-a_2\in\mathbb{N}\land u\in\mathbb{N}\land a_1-a_j\notin\mathbb{Z}\land 3\le j\le n\land a_2-b_j-1\notin\mathbb{N}\land 1\le j\le m\land -a_2+b_j+u\notin\mathbb{N}\land m+1\le j\le q\land -a_j+a_2-u-1\notin\mathbb{N}\land n+1\le j\le p$$

```
Assuming[a_2 = a_1 - h,
  (Table[{Residue[(z^-s (∏_{k=1}^m Gamma[s+b_k]) ∏_{k=1}^n Gamma[1-s-a_k]) / ((∏_{k=n+1}^p Gamma[s+a_k]) ∏_{k=m+1}^q Gamma[1-s-b_k]), {s, 1 - a_2 + u}, Assumptions → {u ∈ Integers ∧ u ≥ 0 ∧
        h ∈ Integers ∧ h ≥ 0}] / (((-1)^(-1+a_1-a_2) (∏_{k=1}^m Gamma[1+u-a_2+b_k]) ∏_{k=3}^n Gamma[-u+a_2-a_k]) / (u! (u+a_1-a_2)! (∏_{k=n+1}^p Gamma[1+u-a_2+a_k]) ∏_{k=m+1}^q Gamma[-u+a_2-b_k])
      z^(-1-u+a_2) (Log[z] + PolyGamma[0, 1 + u] + PolyGamma[0, 1 + u + a_1 - a_2] +
        ∑_{k=3}^n PolyGamma[0, -u + a_2 - a_k] + ∑_{k=n+1}^p PolyGamma[0, 1 + u - a_2 + a_k] -
        ∑_{k=m+1}^q PolyGamma[0, -u + a_2 - b_k] - ∑_{k=1}^m PolyGamma[0, 1 + u - a_2 + b_k]))}, {m, 0, 2},
    {n, 2, 3}, {p, n, 4}, {q, m, 4}] // Simplify[#, Assumptions → {k ∈ Integers}] &)] //
  Simplify[#, Assumptions → {u ∈ Integers ∧ u ≥ 0 ∧ h ∈ Integers ∧ h ≥ 0}] &
Assuming[a_2 = a_1 - h,
  (Table[{Residue[(z^-s (∏_{k=1}^m Gamma[s+b_k]) ∏_{k=1}^n Gamma[1-s-a_k]) / ((∏_{k=n+1}^p Gamma[s+a_k]) ∏_{k=m+1}^q Gamma[1-s-b_k]), {s, 1 - a_2 + u}, Assumptions → {u ∈ Integers ∧ u ≥ 0 ∧
        h ∈ Integers ∧ h ≥ 0}] / (GammaResidueRight[{Table[b_i, {i, m}], Table[1 - a_i, {i, n}]},
      {Table[a_i, {i, n + 1, p}], Table[1 - b_i, {i, m + 1, q}]}, {1 - a_2, 2, u}, z])}, {m, 0, 2},
    {n, 2, 3}, {p, n, 4}, {q, m, 4}] // Simplify[#, Assumptions → {k ∈ Integers}] &)] //
  Simplify[#, Assumptions → {u ∈ Integers ∧ u ≥ 0 ∧ h ∈ Integers ∧ h ≥ 0}] &
```

Case of left triple poles

```
Residue[(∏_{k=1}^m Gamma[s + b_k]) ∏_{k=1}^n Gamma[1 - s - a_k] / ((∏_{k=n+1}^p Gamma[s + a_k]) ∏_{k=m+1}^q Gamma[1 - s - b_k]) z^-s, {s, -b_3 - u}] ==
  (-1)^(u-b_1+b_2) z^(u+b_3) (∏_{k=1}^n Gamma[1 + u - a_k + b_3]) (∏_{k=4}^m Gamma[-u - b_3 + b_k]) /
  (2 u! (u - b_1 + b_3)! (u - b_2 + b_3)! (∏_{k=n+1}^p Gamma[-u + a_k - b_3]) ∏_{k=m+1}^q Gamma[1 + u + b_3 - b_k])
   (π^2 - PolyGamma[1, 1 + u] - PolyGamma[1, 1 + u - b_1 + b_3] - PolyGamma[1, 1 + u - b_2 + b_3] +
     (-Log[z] + PolyGamma[0, 1 + u] + PolyGamma[0, 1 + u - b_1 + b_3] + PolyGamma[0, 1 + u - b_2 + b_3] -
```

$$\sum_{k=n+1}^{p} \text{PolyGamma}[0, -u + a_k - b_3] - \sum_{k=1}^{n} \text{PolyGamma}[0, 1 + u - a_k + b_3] +$$
$$\left.\sum_{k=m+1}^{q} \text{PolyGamma}[0, 1 + u + b_3 - b_k] + \sum_{k=4}^{m} \text{PolyGamma}[0, -u - b_3 + b_k]\right)^2 -$$
$$\sum_{k=n+1}^{p} \text{PolyGamma}[1, -u + a_k - b_3] + \sum_{k=1}^{n} \text{PolyGamma}[1, 1 + u - a_k + b_3] -$$
$$\left.\sum_{k=m+1}^{q} \text{PolyGamma}[1, 1 + u + b_3 - b_k] + \sum_{k=4}^{m} \text{PolyGamma}[1, -u - b_3 + b_k]\right) / ;$$

$-b_2 + b_3 \in \text{Integers} \&\& -b_2 + b_3 \geq 0 \&\& -b_1 + b_2 \in \text{Integers} \&\&$
$-b_1 + b_2 \geq 0 \&\&$
$u \in \text{Integers} \&\&$
$u \geq 0 \&\&$
$-b_1 + b_j \notin \text{Integers} \&\&$
$4 \leq j \leq m \&\&$
$!\left(a_j - b_3 \in \text{Integers} \&\& -1 + a_j - b_3 \geq 0\right) \&\&$
$1 \leq j \leq n \&\&$
$!\left(u - a_j + b_3 \in \text{Integers} \&\& u - a_j + b_3 \geq 0\right) \&\&$
$n + 1 \leq j \leq p \&\&$
$!\left(-u - b_3 + b_j \in \text{Integers} \&\& -1 - u - b_3 + b_j \geq 0\right) \&\& m + 1 \leq j \leq q$

$$\operatorname{res}\left(\frac{\left(\prod_{k=1}^{m}\Gamma(s+b_k)\right)\prod_{k=1}^{n}\Gamma(-s-a_k+1)}{\left(\prod_{k=n+1}^{p}\Gamma(s+a_k)\right)\prod_{k=m+1}^{q}\Gamma(-s-b_k+1)}\, z^{-s}, \{s, -b_3-u\}\right) =$$
$$\frac{(-1)^{-b_1+b_2+u}\left(\prod_{k=4}^{m}\Gamma(-u-b_3+b_k)\right)\prod_{k=1}^{n}\Gamma(u-a_k+b_3+1)}{2\, u!\,(-b_1+b_3+u)!\,(-b_2+b_3+u)!\,\prod_{k=m+1}^{q}\Gamma(u+b_3-b_k+1)\left(\prod_{k=n+1}^{p}\Gamma(-u+a_k-b_3)\right)}\, z^{b_3+u}$$
$$\left(\left(-\sum_{k=n+1}^{p}\psi^{(0)}(-u+a_k-b_3) - \sum_{k=1}^{n}\psi^{(0)}(u-a_k+b_3+1) + \sum_{k=m+1}^{q}\psi^{(0)}(u+b_3-b_k+1) + \right.\right.$$
$$\left.\sum_{k=4}^{m}\psi^{(0)}(-u-b_3+b_k) + \psi^{(0)}(u-b_1+b_3+1) + \psi^{(0)}(u-b_2+b_3+1) + \psi^{(0)}(u+1) - \log(z)\right)^2 -$$
$$\sum_{k=n+1}^{p}\psi^{(1)}(-u+a_k-b_3) + \sum_{k=1}^{n}\psi^{(1)}(u-a_k+b_3+1) - \sum_{k=m+1}^{q}\psi^{(1)}(u+b_3-b_k+1) + \sum_{k=4}^{m}\psi^{(1)}(-u-b_3+b_k) -$$
$$\left.\psi^{(1)}(u-b_1+b_3+1) - \psi^{(1)}(u-b_2+b_3+1) - \psi^{(1)}(u+1) + \pi^2\right) /;$$

$b_3 - b_2 \in \mathbb{N} \wedge b_2 - b_1 \in \mathbb{N} \wedge u \in \mathbb{N} \wedge b_j - b_1 \notin \mathbb{Z} \wedge 4 \leq j \leq m \wedge a_j - b_3 - 1 \notin \mathbb{N} \wedge 1 \leq j \leq n \wedge$
$-a_j + b_3 + u \notin \mathbb{N} \wedge n + 1 \leq j \leq p \wedge b_j - b_3 - u - 1 \notin \mathbb{N} \wedge m + 1 \leq j \leq q$

```
Assuming[b_2 = b_1 + h_1,
  Assuming[b_3 = b_1 + h_1 + h_2, (Table[{Residue[((∏_{k=1}^{n} Gamma[1-s-a_k]) ∏_{k=1}^{m} Gamma[s+b_k]) / ((∏_{k=n+1}^{p} Gamma[s+a_k]) ∏_{k=m+1}^{q} Gamma[1-s-b_k]) z^(-s), {s, -b_3 - u},
          Assumptions→ {u ∈ Integers ∧ u ≥ 0 ∧ h_1 ∈ Integers ∧ h_1 ≥ 0 ∧ h_2 ∈ Integers ∧ h_2 ≥ 0}] /
        ((-1)^(u-b_1+b_2) z^(u+b_3) (∏_{k=1}^{n} Gamma[1+u-a_k+b_3]) (∏_{k=4}^{m} Gamma[-u-b_3+b_k]) / (2 u! (u-b_1+b_3)! (u-b_2+b_3)! (∏_{k=n+1}^{p} Gamma[-u+a_k-b_3]) ∏_{k=m+1}^{q} Gamma[1+u+b_3-b_k])
          (π^2 - PolyGamma[1, 1 + u] - PolyGamma[1, 1 + u - b_1 + b_3] -
            PolyGamma[1, 1 + u - b_2 + b_3] + (-Log[z] + PolyGamma[0, 1 + u] +
                PolyGamma[0, 1 + u - b_1 + b_3] + PolyGamma[0, 1 + u - b_2 + b_3] -
                ∑_{k=n+1}^{p} PolyGamma[0, -u + a_k - b_3] - ∑_{k=1}^{n} PolyGamma[0, 1 + u - a_k + b_3] +
                ∑_{k=m+1}^{q} PolyGamma[0, 1 + u + b_3 - b_k] + ∑_{k=4}^{m} PolyGamma[0, -u - b_3 + b_k])^2 -
            ∑_{k=n+1}^{p} PolyGamma[1, -u + a_k - b_3] + ∑_{k=1}^{n} PolyGamma[1, 1 + u - a_k + b_3] -
            ∑_{k=m+1}^{q} PolyGamma[1, 1 + u + b_3 - b_k] + ∑_{k=4}^{m} PolyGamma[1, -u - b_3 + b_k]))},
      {m, 3, 5}, {n, 0, 2}, {p, n, 4}, {q, m, 4}] // Simplify[#,
        Assumptions→ {k ∈ Integers}] &) //
    Simplify[#, Assumptions→ {u ∈ Integers ∧ u ≥ 0 ∧ h_1 ∈ Integers ∧ h_1 ≥ 0 ∧ h_2 ∈ Integers ∧ h_2 ≥ 0}] &]] //
 Simplify[#,
   Assumptions→
    {u ∈ Integers ∧
      u ≥ 0 ∧
      h_1 ∈ Integers ∧
      h_1 ≥ 0 ∧
      h_2 ∈ Integers ∧
      h_2 ≥ 0}] &
Assuming[b_2 = b_1 + h_1,
  Assuming[b_3 = b_1 + h_1 + h_2, (Table[{Residue[((∏_{k=1}^{n} Gamma[1-s-a_k]) ∏_{k=1}^{m} Gamma[s+b_k]) / ((∏_{k=n+1}^{p} Gamma[s+a_k]) ∏_{k=m+1}^{q} Gamma[1-s-b_k]) z^(-s), {s, -b_3 - u},
          Assumptions→ {u ∈ Integers ∧ u ≥ 0 ∧ h_1 ∈ Integers ∧ h_1 ≥ 0 ∧ h_2 ∈ Integers ∧ h_2 ≥ 0}] /
        (GammaResidueLeft[{Table[b_i, {i, m}], Table[1 - a_i, {i, n}]},
          {Table[a_i, {i, n + 1, p}], Table[1 - b_i, {i, m + 1, q}]}, {b_3, 3, u}, z])}, {m, 3, 5},
      {n, 0, 2}, {p, n, 4}, {q, m, 4}] // Simplify[#, Assumptions→ {k ∈ Integers}] &) //
    Simplify[#, Assumptions→ {u ∈ Integers ∧ u ≥ 0 ∧ h_1 ∈ Integers ∧ h_1 ≥ 0 ∧ h_2 ∈ Integers ∧ h_2 ≥ 0}] &]] //
 Simplify[#, Assumptions→ {u ∈ Integers ∧ u ≥ 0 ∧ h_1 ∈ Integers ∧ h_1 ≥ 0 ∧ h_2 ∈ Integers ∧ h_2 ≥ 0}] &
Residue[((∏_{k=1}^{m} Gamma[s + b_k]) ∏_{k=1}^{n} Gamma[1 - s - a_k]) / ((∏_{k=n+1}^{p} Gamma[s + a_k]) ∏_{k=m+1}^{q} Gamma[1 - s - b_k]) z^(-s), {s, -b_3 - u}] ==
 GammaResidueLeft[{Table[b_i, {i, m}], Table[1 - a_i, {i, n}]},
```

```
{Table[a_i, {i, n + 1, p}], Table[1 - b_i, {i, m + 1, q}]}, {b_3, 3, u}, z] /;
-b_2 + b_3 ∈ Integers && -b_2 + b_3 ≥ 0 && -b_1 + b_2 ∈ Integers && -b_1 + b_2 ≥ 0 && u ∈ Integers &&
 u ≥ 0 && -b_1 + b_j ∉ Integers && 4 ≤ j ≤ m && ! (a_j - b_3 ∈ Integers && -1 + a_j - b_3 ≥ 0) &&
 1 ≤ j ≤ n && ! (u - a_j + b_3 ∈ Integers && u - a_j + b_3 ≥ 0) && n + 1 ≤ j ≤ p &&
 ! (-u - b_3 + b_j ∈ Integers && -1 - u - b_3 + b_j ≥ 0) && m + 1 ≤ j ≤ q
```

$$\operatorname{res}\left(\frac{\left(\prod_{k=1}^{m}\Gamma(s+b_k)\right)\prod_{k=1}^{n}\Gamma(-s-a_k+1)}{\left(\prod_{k=n+1}^{p}\Gamma(s+a_k)\right)\prod_{k=m+1}^{q}\Gamma(-s-b_k+1)}\,z^{-s},\ \{s,\ -b_3-u\}\right) = \text{GammaResidueLeft}($$

$$\{\text{Table}[b_i, \{i, m\}], \text{Table}[1-a_i, \{i, n\}]\}, \{\text{Table}[a_i, \{i, n+1, p\}], \text{Table}[1-b_i, \{i, m+1, q\}]\}, \{b_3, 3, u\}, z)\ /;$$
$$b_3 - b_2 \in \mathbb{N} \land b_2 - b_1 \in \mathbb{N} \land u \in \mathbb{N} \land b_j - b_1 \notin \mathbb{Z} \land 4 \le j \le m \land a_j - b_3 - 1 \notin \mathbb{N} \land 1 \le j \le n \land$$
$$-a_j + b_3 + u \notin \mathbb{N} \land n+1 \le j \le p \land b_j - b_3 - u - 1 \notin \mathbb{N} \land m+1 \le j \le q$$

```
Assuming[b_2 = b_1 + h_1,
 Assuming[b_3 = b_1 + h_1 + h_2, (Table[{Residue[((∏_{k=1}^n Gamma[1-s-a_k]) ∏_{k=1}^m Gamma[s+b_k]) / ((∏_{k=n+1}^p Gamma[s+a_k]) ∏_{k=m+1}^q Gamma[1-s-b_k]) z^-s, {s, -b_3 - u},
        Assumptions → {u ∈ Integers ∧ u ≥ 0 ∧ h_1 ∈ Integers ∧ h_1 ≥ 0 ∧ h_2 ∈ Integers ∧ h_2 ≥ 0}] /
       (((-1)^(u-b_1+b_2) z^(u+b_3) (∏_{k=1}^n Gamma[1+u-a_k+b_3]) (∏_{k=4}^m Gamma[-u-b_3+b_k])) / (2 u! (u-b_1+b_3)! (u-b_2+b_3)! (∏_{k=n+1}^p Gamma[-u+a_k-b_3]) ∏_{k=m+1}^q Gamma[1+u+b_3-b_k])
         (π^2 - PolyGamma[1, 1 + u] - PolyGamma[1, 1 + u - b_1 + b_3] -
           PolyGamma[1, 1 + u - b_2 + b_3] + (-Log[z] + PolyGamma[0, 1 + u] +
               PolyGamma[0, 1 + u - b_1 + b_3] + PolyGamma[0, 1 + u - b_2 + b_3] -
               ∑_{k=n+1}^p PolyGamma[0, -u + a_k - b_3] - ∑_{k=1}^n PolyGamma[0, 1 + u - a_k + b_3] +
               ∑_{k=m+1}^q PolyGamma[0, 1 + u + b_3 - b_k] + ∑_{k=4}^m PolyGamma[0, -u - b_3 + b_k])^2 -
           ∑_{k=n+1}^p PolyGamma[1, -u + a_k - b_3] + ∑_{k=1}^n PolyGamma[1, 1 + u - a_k + b_3] -
           ∑_{k=m+1}^q PolyGamma[1, 1 + u + b_3 - b_k] + ∑_{k=4}^m PolyGamma[1, -u - b_3 + b_k]))},
      {m, 3, 5}, {n, 0, 2}, {p, n, 4}, {q, m, 4}] // Simplify[#,
       Assumptions → {k ∈ Integers}] &) //
   Simplify[#, Assumptions → {u ∈ Integers ∧ u ≥ 0 ∧ h_1 ∈ Integers ∧ h_1 ≥ 0 ∧ h_2 ∈ Integers ∧ h_2 ≥ 0}] &]] //
 Simplify[#,
   Assumptions →
    {u ∈ Integers ∧
      u ≥ 0 ∧
      h_1 ∈ Integers ∧
      h_1 ≥ 0 ∧
      h_2 ∈ Integers ∧
      h_2 ≥ 0}] &
```

```
Assuming[b₂ = b₁ + h₁,
  Assuming[b₃ = b₁ + h₁ + h₂, (Table[{Residue[((∏_{k=1}^{n} Gamma[1-s-a_k]) ∏_{k=1}^{m} Gamma[s+b_k]) / ((∏_{k=n+1}^{p} Gamma[s+a_k]) ∏_{k=m+1}^{q} Gamma[1-s-b_k]) z^-s, {s, -b₃ - u},
        Assumptions→ {u ∈ Integers ∧ u ≥ 0 ∧ h₁ ∈ Integers ∧ h₁ ≥ 0 ∧ h₂ ∈ Integers ∧ h₂ ≥ 0}] /
       (GammaResidueLeft[{Table[b_i, {i, m}], Table[1 - a_i, {i, n}]},
          {Table[a_i, {i, n + 1, p}], Table[1 - b_i, {i, m + 1, q}]}, {b₃, 3, u}, z])}, {m, 3, 5},
      {n, 0, 2}, {p, n, 4}, {q, m, 4}] // Simplify[#, Assumptions→ {k ∈ Integers}] &) //
    Simplify[#, Assumptions→ {u ∈ Integers ∧ u ≥ 0 ∧ h₁ ∈ Integers ∧ h₁ ≥ 0 ∧ h₂ ∈ Integers ∧ h₂ ≥ 0}] &]] //
  Simplify[#, Assumptions→ {u ∈ Integers ∧ u ≥ 0 ∧ h₁ ∈ Integers ∧ h₁ ≥ 0 ∧ h₂ ∈ Integers ∧ h₂ ≥ 0}] &
```

Case of right triple poles

$$\text{Residue}\left[\frac{\left(\prod_{k=1}^{m}\text{Gamma}[s+b_k]\right)\prod_{k=1}^{n}\text{Gamma}[1-s-a_k]}{\left(\prod_{k=n+1}^{p}\text{Gamma}[s+a_k]\right)\prod_{k=m+1}^{q}\text{Gamma}[1-s-b_k]}z^{-s},\{s,1-a_3+u\}\right]==$$

$$\frac{(-1)^{-1+u+a_1-a_2}\left(\prod_{k=1}^{m}\text{Gamma}[1+u-a_3+b_k]\right)\prod_{k=4}^{n}\text{Gamma}[-u+a_3-a_k]}{2\,u!\,(u+a_1-a_3)!\,(u+a_2-a_3)!\left(\prod_{k=n+1}^{p}\text{Gamma}[1+u-a_3+a_k]\right)\prod_{k=m+1}^{q}\text{Gamma}[-u+a_3-b_k]}z^{-1-u+a_3}$$
$$\Bigg(\pi^2-\text{PolyGamma}[1,1+u]-\text{PolyGamma}[1,1+u+a_1-a_3]-\text{PolyGamma}[1,1+u+a_2-a_3]+$$
$$\Bigg(\text{Log}[z]+\text{PolyGamma}[0,1+u]+\text{PolyGamma}[0,1+u+a_1-a_3]+\text{PolyGamma}[0,1+u+a_2-a_3]+$$
$$\sum_{k=4}^{n}\text{PolyGamma}[0,-u+a_3-a_k]+\sum_{k=n+1}^{p}\text{PolyGamma}[0,1+u-a_3+a_k]-$$
$$\sum_{k=m+1}^{q}\text{PolyGamma}[0,-u+a_3-b_k]-\sum_{k=1}^{m}\text{PolyGamma}[0,1+u-a_3+b_k]\Bigg)^2+$$
$$\sum_{k=4}^{n}\text{PolyGamma}[1,-u+a_3-a_k]-\sum_{k=n+1}^{p}\text{PolyGamma}[1,1+u-a_3+a_k]-$$
$$\sum_{k=m+1}^{q}\text{PolyGamma}[1,-u+a_3-b_k]+\sum_{k=1}^{m}\text{PolyGamma}[1,1+u-a_3+b_k]\Bigg)\;/;$$

$a_2 - a_3 \in \text{Integers} \&\& a_2 - a_3 \ge 0 \&\& a_1 - a_2 \in \text{Integers} \&\&$
$a_1 - a_2 \ge 0 \&\&$
$u \in \text{Integers} \&\&$
$u \ge 0 \&\&$
$a_1 - a_j \notin \text{Integers} \&\&$
$4 \le j \le n \&\&$
$!\left(a_3 - b_j \in \text{Integers} \&\& -1 + a_3 - b_j \ge 0\right) \&\&$
$1 \le j \le m \&\&$
$!\left(u - a_3 + b_j \in \text{Integers} \&\& u - a_3 + b_j \ge 0\right) \&\&$

m + 1 ≤ j ≤ q &&
! (−u + a_3 − a_j ∈ Integers && −1 − u + a_3 − a_j ≥ 0) && n + 1 ≤ j ≤ p

$$\operatorname{res}_s\left(\frac{\left(\prod_{k=1}^{m}\Gamma(s+b_k)\right)\prod_{k=1}^{n}\Gamma(-s-a_k+1)}{\left(\prod_{k=n+1}^{p}\Gamma(s+a_k)\right)\prod_{k=m+1}^{q}\Gamma(-s-b_k+1)}\,z^{-s}\right)(1-a_3+u)==$$

$$\frac{(-1)^{a_1-a_2+u-1}\left(\prod_{k=1}^{m}\Gamma(u-a_3+b_k+1)\right)\prod_{k=4}^{n}\Gamma(-u+a_3-a_k)}{2\,u!\,(a_1-a_3+u)!\,(a_2-a_3+u)!\left(\prod_{k=n+1}^{p}\Gamma(u-a_3+a_k+1)\right)\prod_{k=m+1}^{q}\Gamma(-u+a_3-b_k)}$$
$$z^{a_3-u-1}\Bigg(\Bigg(-\sum_{k=m+1}^{q}\psi^{(0)}(-u+a_3-b_k)-\sum_{k=1}^{m}\psi^{(0)}(u-a_3+b_k+1)+\sum_{k=n+1}^{p}\psi^{(0)}(u-a_3+a_k+1)+$$
$$\sum_{k=4}^{n}\psi^{(0)}(-u+a_3-a_k)+\psi^{(0)}(u+a_1-a_3+1)+\psi^{(0)}(u+a_2-a_3+1)+\psi^{(0)}(u+1)+\log(z)\Bigg)^2-$$
$$\sum_{k=m+1}^{q}\psi^{(1)}(-u+a_3-b_k)+\sum_{k=1}^{m}\psi^{(1)}(u-a_3+b_k+1)-\sum_{k=n+1}^{p}\psi^{(1)}(u-a_3+a_k+1)+\sum_{k=4}^{n}\psi^{(1)}(-u+a_3-a_k)-$$
$$\psi^{(1)}(u+a_1-a_3+1)-\psi^{(1)}(u+a_2-a_3+1)-\psi^{(1)}(u+1)+\pi^2\Bigg)\,/;$$

$$a_2-a_3\in\mathbb{N}\land a_1-a_2\in\mathbb{N}\land u\in\mathbb{N}\land a_1-a_j\notin\mathbb{Z}\land 4\le j\le n\land a_3-b_j-1\notin\mathbb{N}\land 1\le j\le m\land$$
$$-a_3+b_j+u\notin\mathbb{N}\land m+1\le j\le q\land$$
$$-a_j+a_3-u-1\notin\mathbb{N}\land n+1\le j\le p$$

```
Assuming[a2 = a1 − h1,
  Assuming[a3 = a1 − h1 − h2, (Table[{Residue[ z^−s (∏_{k=1}^m Gamma[s+bk]) ∏_{k=1}^n Gamma[1−s−ak] / ((∏_{k=n+1}^p Gamma[s+ak]) ∏_{k=m+1}^q Gamma[1−s−bk]), {s, 1 − a3 + u},
        Assumptions→ {u ∈ Integers ∧ u ≥ 0 ∧ h1 ∈ Integers ∧ h1 ≥ 0 ∧ h2 ∈ Integers ∧ h2 ≥ 0} ] /
      ((−1)^(−1+u+a1−a2) (∏_{k=1}^m Gamma[1+u−a3+bk]) ∏_{k=4}^n Gamma[−u+a3−ak] / (2 u! (u+a1−a3)! (u+a2−a3)! (∏_{k=n+1}^p Gamma[1+u−a3+ak]) ∏_{k=m+1}^q Gamma[−u+a3−bk]) z^(−1−u+a3)
        (π^2 − PolyGamma[1, 1 + u] − PolyGamma[1, 1 + u + a1 − a3] −
          PolyGamma[1, 1 + u + a2 − a3] + (Log[z] + PolyGamma[0, 1 + u] +
              PolyGamma[0, 1 + u + a1 − a3] + PolyGamma[0, 1 + u + a2 − a3] +
              ∑_{k=4}^n PolyGamma[0, −u + a3 − ak] + ∑_{k=n+1}^p PolyGamma[0, 1 + u − a3 + ak] −
              ∑_{k=m+1}^q PolyGamma[0, −u + a3 − bk] − ∑_{k=1}^m PolyGamma[0, 1 + u − a3 + bk])^2 +
          ∑_{k=4}^n PolyGamma[1, −u + a3 − ak] − ∑_{k=n+1}^p PolyGamma[1, 1 + u − a3 + ak] −
          ∑_{k=m+1}^q PolyGamma[1, −u + a3 − bk] + ∑_{k=1}^m PolyGamma[1, 1 + u − a3 + bk]))}, {m, 0, 2},
      {n, 3, 4}, {p, n, 5}, {q, m, 5}] // Simplify[#, Assumptions→ {k ∈ Integers}] &)]] //
 Simplify[#, Assumptions→ {u ∈ Integers ∧ u ≥ 0 ∧ h1 ∈ Integers ∧
      h1 ≥ 0 ∧
      h2 ∈ Integers ∧ h2 ≥ 0}] &
```

```
Assuming[a2 = a1 - h1,
  Assuming[a3 = a1 - h1 - h2, (Table[{Residue[(z^-s (∏_{k=1}^m Gamma[s+bk]) ∏_{k=1}^n Gamma[1-s-ak]) / ((∏_{k=n+1}^p Gamma[s+ak]) ∏_{k=m+1}^q Gamma[1-s-bk]), {s, 1 - a3 + u},
          Assumptions→ {u ∈ Integers∧ u ≥ 0 ∧ h1 ∈ Integers∧ h1 ≥ 0 ∧ h2 ∈ Integers∧ h2 ≥ 0}] /
         (GammaResidueRight[ {Table[bi, {i, m}], Table[1 - ai, {i, n}]},
            {Table[ai, {i, n + 1, p}], Table[1 - bi, {i, m + 1, q}]}, {1 - a3, 3, u}, z])}, {m, 0, 2},
        {n, 3, 4}, {p, n, 5}, {q, m, 5}] // Simplify[#, Assumptions→ {k ∈ Integers}] &)]] //
  Simplify[#, Assumptions→ {u ∈ Integers∧ u ≥ 0 ∧ h1 ∈ Integers∧ h1 ≥ 0 ∧ h2 ∈ Integers∧ h2 ≥ 0}] &
Residue[((∏_{k=1}^m Gamma[s + bk]) ∏_{k=1}^n Gamma[1 - s - ak]) / ((∏_{k=n+1}^p Gamma[s + ak]) ∏_{k=m+1}^q Gamma[1 - s - bk]) z^-s, {s, 1 - a3 + u}] ==
   GammaResidueRight[ {Table[bi, {i, m}], Table[1 - ai, {i, n}]},
    {Table[ai, {i, n + 1, p}], Table[1 - bi, {i, m + 1, q}]}, {1 - a3, 3, u}, z] /;
  a2 - a3 ∈ Integers&& a2 - a3 ≥ 0 && a1 - a2 ∈ Integers&& a1 - a2 ≥ 0 && u ∈ Integers&& u ≥ 0 &&
   a1 - aj ∉ Integers&& 4 ≤ j ≤ n && ! (a3 - bj ∈ Integers&& -1 + a3 - bj ≥ 0) &&
   1 ≤ j ≤ m && ! (u - a3 + bj ∈ Integers&& u - a3 + bj ≥ 0) && m + 1 ≤ j ≤ q &&
   ! (-u + a3 - aj ∈ Integers&& -1 - u + a3 - aj ≥ 0) && n + 1 ≤ j ≤ p
```

$$\operatorname{res}_s\left(\frac{\left(\prod_{k=1}^{m}\Gamma(s+b_k)\right)\prod_{k=1}^{n}\Gamma(-s-a_k+1)}{\left(\prod_{k=n+1}^{p}\Gamma(s+a_k)\right)\prod_{k=m+1}^{q}\Gamma(-s-b_k+1)}\,z^{-s}\right)(1-a_3+u) == \text{GammaResidueRight}(\{\text{Table}[b_i,\{i,m\}],\ \text{Table}[1-a_i,\{i,n\}]\},\ \{\text{Table}[a_i,\{i,n+1,p\}],\ \text{Table}[1-b_i,\{i,m+1,q\}]\},\ \{1-a_3,3,u\},\ z)\ /;$$
$$a_2-a_3\in\mathbb{N}\land a_1-a_2\in\mathbb{N}\land u\in\mathbb{N}\land a_1-a_j\notin\mathbb{Z}\land 4\le j\le n\land a_3-b_j-1\notin\mathbb{N}\land 1\le j\le m\land -a_3+b_j+u\notin\mathbb{N}\land m+1\le j\le q\land -a_j+a_3-u-1\notin\mathbb{N}\land n+1\le j\le p$$

```
Assuming[a₂ = a₁ - h₁,
  Assuming[a₃ = a₁ - h₁ - h₂, (Table[{Residue[z^-s (∏_{k=1}^m Gamma[s+b_k]) ∏_{k=1}^n Gamma[1-s-a_k] / ((∏_{k=n+1}^p Gamma[s+a_k]) ∏_{k=m+1}^q Gamma[1-s-b_k]), {s, 1 - a₃ + u},
        Assumptions→ {u ∈ Integers ∧ u ≥ 0 ∧ h₁ ∈ Integers ∧ h₁ ≥ 0 ∧ h₂ ∈ Integers ∧ h₂ ≥ 0}] /
      ((-1)^(-1+u+a₁-a₂) (∏_{k=1}^m Gamma[1+u-a₃+b_k]) ∏_{k=4}^n Gamma[-u+a₃-a_k] / (2 u! (u+a₁-a₃)! (u+a₂-a₃)! (∏_{k=n+1}^p Gamma[1+u-a₃+a_k]) ∏_{k=m+1}^q Gamma[-u+a₃-b_k]) z^(-1-u+a₃)
        (π² - PolyGamma[1, 1 + u] - PolyGamma[1, 1 + u + a₁ - a₃] -
          PolyGamma[1, 1 + u + a₂ - a₃] + (Log[z] + PolyGamma[0, 1 + u] +
              PolyGamma[0, 1 + u + a₁ - a₃] + PolyGamma[0, 1 + u + a₂ - a₃] +
              ∑_{k=4}^n PolyGamma[0, -u + a₃ - a_k] + ∑_{k=n+1}^p PolyGamma[0, 1 + u - a₃ + a_k] -
              ∑_{k=m+1}^q PolyGamma[0, -u + a₃ - b_k] - ∑_{k=1}^m PolyGamma[0, 1 + u - a₃ + b_k])² +
          ∑_{k=4}^n PolyGamma[1, -u + a₃ - a_k] - ∑_{k=n+1}^p PolyGamma[1, 1 + u - a₃ + a_k] -
          ∑_{k=m+1}^q PolyGamma[1, -u + a₃ - b_k] + ∑_{k=1}^m PolyGamma[1, 1 + u - a₃ + b_k]))}, {m, 0, 2},
     {n, 3, 4}, {p, n, 5}, {q, m, 5}] // Simplify[#, Assumptions→ {k ∈ Integers}] &)]] //
  Simplify[#, Assumptions→ {u ∈ Integers ∧ u ≥ 0 ∧ h₁ ∈ Integers ∧
          h₁ ≥ 0 ∧
          h₂ ∈ Integers ∧ h₂ ≥ 0}] &
Assuming[a₂ = a₁ - h₁,
  Assuming[a₃ = a₁ - h₁ - h₂, (Table[{Residue[z^-s (∏_{k=1}^m Gamma[s+b_k]) ∏_{k=1}^n Gamma[1-s-a_k] / ((∏_{k=n+1}^p Gamma[s+a_k]) ∏_{k=m+1}^q Gamma[1-s-b_k]), {s, 1 - a₃ + u},
        Assumptions→ {u ∈ Integers ∧ u ≥ 0 ∧ h₁ ∈ Integers ∧ h₁ ≥ 0 ∧ h₂ ∈ Integers ∧ h₂ ≥ 0}] /
      (GammaResidueRight[{Table[b_i, {i, m}], Table[1 - a_i, {i, n}]},
          {Table[a_i, {i, n + 1, p}], Table[1 - b_i, {i, m + 1, q}]}, {1 - a₃, 3, u}, z])}, {m, 0, 2},
     {n, 3, 4}, {p, n, 5}, {q, m, 5}] // Simplify[#, Assumptions→ {k ∈ Integers}] &)]] //
  Simplify[#, Assumptions→ {u ∈ Integers ∧ u ≥ 0 ∧ h₁ ∈ Integers ∧ h₁ ≥ 0 ∧ h₂ ∈ Integers ∧ h₂ ≥ 0}] &
```

Case of left quartic poles

$$\text{Residue}\left[\frac{\left(\prod_{k=1}^{m}\text{Gamma}[s+b_k]\right)\prod_{k=1}^{n}\text{Gamma}[1-s-a_k]}{\left(\prod_{k=n+1}^{p}\text{Gamma}[s+a_k]\right)\prod_{k=m+1}^{q}\text{Gamma}[1-s-b_k]}\, z^{-s},\ \{s, -b_4-u\}\right] ==$$

$$\left(\left((-1)^{-b_1+b_2-b_3+b_4}\left(\prod_{k=5}^{m}\text{Gamma}[-u-b_4+b_k]\right)\prod_{k=1}^{n}\text{Gamma}[1+u-a_k+b_4]\right)\Big/\left(6\,u!\ (u-b_1+b_4)!\ (u-b_2+b_4)!\ (u-b_3+b_4)!\left(\prod_{k=1}^{p}\text{Gamma}[-u+a_k-b_4]\right)\prod_{k=1}^{q}\text{Gamma}[1+u+b_4-b_k]\right)\right) z^{u+b_4}$$

$$\Big(\text{PolyGamma}[2, 1+u] + \text{PolyGamma}[2, 1+u-b_1+b_4] + \text{PolyGamma}[2, 1+u-b_2+b_4] +$$
$$\text{PolyGamma}[2, 1+u-b_3+b_4] +$$

$$\left(-\mathrm{Log}[z] + \mathrm{PolyGamma}[0, 1+u] + \mathrm{PolyGamma}[0, 1+u-b_1+b_4] + \mathrm{PolyGamma}[0, 1+u-b_2+b_4] + \mathrm{PolyGamma}[0, 1+u-b_3+b_4] - \sum_{k=n+1}^{p} \mathrm{PolyGamma}[0, -u+a_k-b_4] - \sum_{k=1}^{n} \mathrm{PolyGamma}[0, 1+u-a_k+b_4] + \sum_{k=m+1}^{q} \mathrm{PolyGamma}[0, 1+u+b_4-b_k] + \sum_{k=5}^{m} \mathrm{PolyGamma}[0, -u-b_4+b_k]\right)^3 +$$

$$\left(-\mathrm{Log}[z] + \mathrm{PolyGamma}[0, 1+u] + \mathrm{PolyGamma}[0, 1+u-b_1+b_4] + \mathrm{PolyGamma}[0, 1+u-b_2+b_4] + \mathrm{PolyGamma}[0, 1+u-b_3+b_4] - \sum_{k=n+1}^{p} \mathrm{PolyGamma}[0, -u+a_k-b_4] - \sum_{k=1}^{n} \mathrm{PolyGamma}[0, 1+u-a_k+b_4] + \sum_{k=m+1}^{q} \mathrm{PolyGamma}[0, 1+u+b_4-b_k] + \sum_{k=5}^{m} \mathrm{PolyGamma}[0, -u-b_4+b_k]\right)$$

$$\left(4\pi^2 + 3\left(-\mathrm{PolyGamma}[1, 1+u] - \mathrm{PolyGamma}[1, 1+u-b_1+b_4] - \mathrm{PolyGamma}[1, 1+u-b_2+b_4] - \mathrm{PolyGamma}[1, 1+u-b_3+b_4] - \sum_{k=n+1}^{p} \mathrm{PolyGamma}[1, -u+a_k-b_4] + \sum_{k=1}^{n} \mathrm{PolyGamma}[1, 1+u-a_k+b_4] - \sum_{k=m+1}^{q} \mathrm{PolyGamma}[1, 1+u+b_4-b_k] + \sum_{k=5}^{m} \mathrm{PolyGamma}[1, -u-b_4+b_k]\right)\right) -$$

$$\sum_{k=n+1}^{p} \mathrm{PolyGamma}[2, -u+a_k-b_4] - \sum_{k=1}^{n} \mathrm{PolyGamma}[2, 1+u-a_k+b_4] + \sum_{k=m+1}^{q} \mathrm{PolyGamma}[2, 1+u+b_4-b_k] + \sum_{k=5}^{m} \mathrm{PolyGamma}[2, -u-b_4+b_k]\Bigg) / ;$$

$-b_3 + b_4 \in \mathrm{Integers}\,\&\&\, -b_3 + b_4 \geq 0 \,\&\&$
$-b_2 +$
$b_3 \in \mathrm{Integers}\,\&\&$
$-b_2 + b_3 \geq 0 \,\&\&\, -b_1 + b_2 \in \mathrm{Integers}\,\&\&$
$-b_1 + b_2 \geq$
$0 \,\&\&\, u \in$
$\mathrm{Integers}\,\&\&\, u \geq$
$0 \,\&\&\, -b_1 + b_j \notin$
$\mathrm{Integers}\,\&\&\, 5 \leq$
$j \leq$
$m \,\&\&$
$!\,(a_j - b_4 \in \mathrm{Integers}\,\&\&\, -1 + a_j - b_4 \geq 0)\,\&\&$

1 ≤
j ≤
n &&
! (u − a_j + b_4 ∈ Integers&& u − a_j + b_4 ≥ 0) &&
n + 1 ≤
j ≤
p &&
! (−u − b_4 + b_j ∈ Integers&& −1 − u − b_4 + b_j ≥ 0) &&
m + 1 ≤ j ≤ q

$$\operatorname{res}\left(\frac{\left(\prod_{k=1}^{m}\Gamma(s+b_k)\right)\prod_{k=1}^{n}\Gamma(-s-a_k+1)}{\left(\prod_{k=n+1}^{p}\Gamma(s+a_k)\right)\prod_{k=m+1}^{q}\Gamma(-s-b_k+1)}\,z^{-s},\,\{s,\,-b_4-u\}\right)=$$

$$\frac{(-1)^{-b_1+b_2-b_3+b_4}\left(\prod_{k=5}^{m}\Gamma(-u-b_4+b_k)\right)\prod_{k=1}^{n}\Gamma(u-a_k+b_4+1)}{6\,u!\,(-b_1+b_4+u)!\,(-b_2+b_4+u)!\,(-b_3+b_4+u)!\,\prod_{k=1}^{q}\Gamma(u+b_4-b_k+1)\left(\prod_{k=1}^{p}\Gamma(-u+a_k-b_4)\right)}\,z^{b_4+u}$$

$$\left(\left(-\sum_{k=n+1}^{p}\psi^{(0)}(-u+a_k-b_4)-\sum_{k=1}^{n}\psi^{(0)}(u-a_k+b_4+1)+\sum_{k=m+1}^{q}\psi^{(0)}(u+b_4-b_k+1)+\sum_{k=5}^{m}\psi^{(0)}(-u-b_4+b_k)+\right.\right.$$
$$\left.\psi^{(0)}(u-b_1+b_4+1)+\psi^{(0)}(u-b_2+b_4+1)+\psi^{(0)}(u-b_3+b_4+1)+\psi^{(0)}(u+1)-\log(z)\right)^3+$$
$$\left(3\left(-\sum_{k=n+1}^{p}\psi^{(1)}(-u+a_k-b_4)+\sum_{k=1}^{n}\psi^{(1)}(u-a_k+b_4+1)-\sum_{k=m+1}^{q}\psi^{(1)}(u+b_4-b_k+1)+\sum_{k=5}^{m}\psi^{(1)}(-u-b_4+b_k)-\right.\right.$$
$$\left.\left.\psi^{(1)}(u-b_1+b_4+1)-\psi^{(1)}(u-b_2+b_4+1)-\psi^{(1)}(u-b_3+b_4+1)-\psi^{(1)}(u+1)\right)+4\pi^2\right)$$
$$\left(-\sum_{k=n+1}^{p}\psi^{(0)}(-u+a_k-b_4)-\sum_{k=1}^{n}\psi^{(0)}(u-a_k+b_4+1)+\sum_{k=m+1}^{q}\psi^{(0)}(u+b_4-b_k+1)+\sum_{k=5}^{m}\psi^{(0)}(-u-b_4+b_k)+\right.$$
$$\left.\psi^{(0)}(u-b_1+b_4+1)+\psi^{(0)}(u-b_2+b_4+1)+\psi^{(0)}(u-b_3+b_4+1)+\psi^{(0)}(u+1)-\log(z)\right)-$$
$$\sum_{k=n+1}^{p}\psi^{(2)}(-u+a_k-b_4)-\sum_{k=1}^{n}\psi^{(2)}(u-a_k+b_4+1)+\sum_{k=m+1}^{q}\psi^{(2)}(u+b_4-b_k+1)+\sum_{k=5}^{m}\psi^{(2)}(-u-b_4+b_k)+$$
$$\left.\psi^{(2)}(u-b_1+b_4+1)+\psi^{(2)}(u-b_2+b_4+1)+\psi^{(2)}(u-b_3+b_4+1)+\psi^{(2)}(u+1)\right)\,/;$$

$b_4-b_3\in\mathbb{N}\land b_3-b_2\in\mathbb{N}\land b_2-b_1\in\mathbb{N}\land u\in\mathbb{N}\land b_j-b_1\notin\mathbb{Z}\land$
$5\le j\le m\land$
$a_j-b_4-1\notin\mathbb{N}\land 1\le j\le n\land$
$-a_j+b_4+u\notin\mathbb{N}\land n+1\le j\le p\land$
$b_j-b_4-u-1\notin\mathbb{N}\land m+1\le j\le q$

```
Assuming[b_2 = b_1 + h_1, Assuming[b_3 = b_1 + h_1 + h_2, Assuming[b_4 = b_1 + h_1 + h_2 + h_3,
    (Table[{Residue[((∏_{k=1}^n Gamma[1-s-a_k]) ∏_{k=1}^m Gamma[s+b_k]) / ((∏_{k=n+1}^p Gamma[s+a_k]) ∏_{k=m+1}^q Gamma[1-s-b_k]) z^(-s), {s, -b_4 - u}, Assumptions→ {u ∈ Integers∧
                u ≥ 0 ∧ h_1 ∈ Integers∧ h_1 ≥ 0 ∧ h_2 ∈ Integers∧ h_2 ≥ 0 ∧ h_3 ∈ Integers∧ h_3 ≥ 0}] /
          (((-1)^(-b_1+b_2-b_3+b_4) (∏_{k=5}^m Gamma[-u-b_4+b_k]) ∏_{k=1}^n Gamma[1+u-a_k+b_4]) / (6 u! (u-b_1+b_4)! (u-b_2+b_4)! (u-b_3+b_4)! (∏_{k=n+1}^p Gamma[-u+a_k-b_4]) ∏_{k=m+1}^q Gamma[1+u+b_4-b_k]) z^(u+b_4)
             (PolyGamma[2, 1 + u] + PolyGamma[2, 1 + u - b_1 + b_4] +
               PolyGamma[2, 1 + u - b_2 + b_4] + PolyGamma[2, 1 + u - b_3 + b_4] +
               (-Log[z] + PolyGamma[0, 1 + u] + PolyGamma[0, 1 + u - b_1 + b_4] +
                   PolyGamma[0, 1 + u - b_2 + b_4] + PolyGamma[0, 1 + u - b_3 + b_4] -
                   Σ_{k=n+1}^p PolyGamma[0, -u + a_k - b_4] - Σ_{k=1}^n PolyGamma[0, 1 + u - a_k + b_4] +
                   Σ_{k=m+1}^q PolyGamma[0, 1 + u + b_4 - b_k] + Σ_{k=5}^m PolyGamma[0, -u - b_4 + b_k])^3 +
               (-Log[z] + PolyGamma[0, 1 + u] + PolyGamma[0, 1 + u - b_1 + b_4] +
                   PolyGamma[0, 1 + u - b_2 + b_4] + PolyGamma[0, 1 + u - b_3 + b_4] -
                   Σ_{k=n+1}^p PolyGamma[0, -u + a_k - b_4] - Σ_{k=1}^n PolyGamma[0, 1 + u - a_k + b_4] +
                   Σ_{k=m+1}^q PolyGamma[0, 1 + u + b_4 - b_k] + Σ_{k=5}^m PolyGamma[0, -u - b_4 + b_k])
                 (4 π^2 + 3 (-PolyGamma[1, 1 + u] - PolyGamma[1, 1 + u - b_1 + b_4] -
                       PolyGamma[1, 1 + u - b_2 + b_4] - PolyGamma[1, 1 + u - b_3 + b_4] -
                       Σ_{k=n+1}^p PolyGamma[1, -u + a_k - b_4] +
                       Σ_{k=1}^n PolyGamma[1, 1 + u - a_k + b_4] - Σ_{k=m+1}^q PolyGamma[1,
                           1 + u + b_4 - b_k] + Σ_{k=5}^m PolyGamma[1, -u - b_4 + b_k])) -
               Σ_{k=n+1}^p PolyGamma[2, -u + a_k - b_4] - Σ_{k=1}^n PolyGamma[2, 1 + u - a_k + b_4] +
               Σ_{k=m+1}^q PolyGamma[2, 1 + u + b_4 - b_k] + Σ_{k=5}^m PolyGamma[2, -u - b_4 + b_k]))},
        {m, 4, 6}, {n, 0, 2}, {p, n, 5}, {q, m, 6}] // Simplify[
          #,
          Assumptions→ {k ∈ Integers}] &) //
      Simplify[#, Assumptions→ {u ∈ Integers∧ u ≥ 0 ∧ h_1 ∈ Integers∧ h_1 ≥ 0 ∧
              h_2 ∈ Integers∧ h_2 ≥ 0 ∧ h_3 ∈ Integers∧ h_3 ≥ 0}] &]]] //
  Simplify[#, Assumptions→ {u ∈ Integers∧
          u ≥
            0 ∧ h_1 ∈
            Integers∧ h_1 ≥
            0 ∧ h_2 ∈
            Integers∧ h_2 ≥
            0 ∧ h_3 ∈
            Integers∧ h_3 ≥
            0}] &
```

```
Assuming[b_2 = b_1 + h_1, Assuming[b_3 = b_1 + h_1 + h_2, Assuming[b_4 = b_1 + h_1 + h_2 + h_3,
  (Table[{Residue[((∏_{k=1}^{n} Gamma[1-s-a_k]) ∏_{k=1}^{m} Gamma[s+b_k]) / ((∏_{k=n+1}^{p} Gamma[s+a_k]) ∏_{k=m+1}^{q} Gamma[1-s-b_k]) z^-s, {s, -b_4 - u}, Assumptions→ {u ∈ Integers ∧
        u ≥ 0 ∧ h_1 ∈ Integers ∧ h_1 ≥ 0 ∧ h_2 ∈ Integers ∧ h_2 ≥ 0 ∧ h_3 ∈ Integers ∧ h_3 ≥ 0}] /
      (GammaResidueLeft[{Table[b_i, {i, m}], Table[1 - a_i, {i, n}]},
        {Table[a_i, {i, n + 1, p}], Table[1 - b_i, {i, m + 1, q}]}, {b_4, 4, u}, z])}, {m, 4, 6},
     {n, 0, 2}, {p, n, 5}, {q, m, 6}] // Simplify[#, Assumptions→ {k ∈ Integers}] &) //
   Simplify[#, Assumptions→ {u ∈ Integers ∧ u ≥ 0 ∧ h_1 ∈ Integers ∧ h_1 ≥ 0 ∧
       h_2 ∈ Integers ∧ h_2 ≥ 0 ∧ h_3 ∈ Integers ∧ h_3 ≥ 0}] &]]] //
 Simplify[#, Assumptions→ {u ∈ Integers ∧ u ≥ 0 ∧ h_1 ∈ Integers ∧ h_1 ≥ 0 ∧ h_2 ∈ Integers ∧
      h_2 ≥ 0 ∧ h_3 ∈ Integers ∧ h_3 ≥ 0}] &
Residue[((∏_{k=1}^{m} Gamma[s + b_k]) ∏_{k=1}^{n} Gamma[1 - s - a_k]) / ((∏_{k=n+1}^{p} Gamma[s + a_k]) ∏_{k=m+1}^{q} Gamma[1 - s - b_k]) z^-s, {s, -b_4 - u}] ==
  GammaResidueLeft[{Table[b_i, {i, m}], Table[1 - a_i, {i, n}]},
   {Table[a_i, {i, n + 1, p}], Table[1 - b_i, {i, m + 1, q}]}, {b_4, 4, u}, z] /;
 -b_3 + b_4 ∈ Integers && -b_3 + b_4 ≥ 0 && -b_2 + b_3 ∈ Integers && -b_2 + b_3 ≥ 0 && -b_1 + b_2 ∈ Integers &&
  -b_1 + b_2 ≥ 0 && u ∈ Integers && u ≥ 0 && -b_1 + b_j ∉ Integers && 5 ≤ j ≤ m &&
  ! (a_j - b_4 ∈ Integers && -1 + a_j - b_4 ≥ 0) && 1 ≤ j ≤ n && ! (u - a_j + b_4 ∈ Integers && u - a_j + b_4 ≥ 0) &&
  n + 1 ≤ j ≤ p && ! (-u - b_4 + b_j ∈ Integers && -1 - u - b_4 + b_j ≥ 0) && m + 1 ≤ j ≤ q
```

$$\operatorname{res}\left(\frac{\left(\prod_{k=1}^{m}\Gamma(s+b_k)\right)\prod_{k=1}^{n}\Gamma(-s-a_k+1)}{\left(\prod_{k=n+1}^{p}\Gamma(s+a_k)\right)\prod_{k=m+1}^{q}\Gamma(-s-b_k+1)}\,z^{-s},\ \{s,\,-b_4-u\}\right)=$$

$$\frac{(-1)^{-b_1+b_2-b_3+b_4}\left(\prod_{k=5}^{m}\Gamma(-u-b_4+b_k)\right)\prod_{k=1}^{n}\Gamma(u-a_k+b_4+1)}{6\,u!\,(-b_1+b_4+u)!\,(-b_2+b_4+u)!\,(-b_3+b_4+u)!\,\prod_{k=1}^{q}\Gamma(u+b_4-b_k+1)\left(\prod_{k=1}^{p}\Gamma(-u+a_k-b_4)\right)}\,z^{b_4+u}$$

$$\Bigg(\Bigg(\Bigg(-\sum_{k=n+1}^{p}\psi^{(0)}(-u+a_k-b_4)-\sum_{k=1}^{n}\psi^{(0)}(u-a_k+b_4+1)+\sum_{k=m+1}^{q}\psi^{(0)}(u+b_4-b_k+1)+\sum_{k=5}^{m}\psi^{(0)}(-u-b_4+b_k)+$$

$$\psi^{(0)}(u-b_1+b_4+1)+\psi^{(0)}(u-b_2+b_4+1)+\psi^{(0)}(u-b_3+b_4+1)+\psi^{(0)}(u+1)-\log(z)\Bigg)^3+$$

$$\Bigg(3\Bigg(-\sum_{k=n+1}^{p}\psi^{(1)}(-u+a_k-b_4)+\sum_{k=1}^{n}\psi^{(1)}(u-a_k+b_4+1)-\sum_{k=m+1}^{q}\psi^{(1)}(u+b_4-b_k+1)+\sum_{k=5}^{m}\psi^{(1)}(-u-b_4+b_k)-$$

$$\psi^{(1)}(u-b_1+b_4+1)-\psi^{(1)}(u-b_2+b_4+1)-\psi^{(1)}(u-b_3+b_4+1)-\psi^{(1)}(u+1)\Bigg)+4\,\pi^2\Bigg)$$

$$\Bigg(-\sum_{k=n+1}^{p}\psi^{(0)}(-u+a_k-b_4)-\sum_{k=1}^{n}\psi^{(0)}(u-a_k+b_4+1)+\sum_{k=m+1}^{q}\psi^{(0)}(u+b_4-b_k+1)+\sum_{k=5}^{m}\psi^{(0)}(-u-b_4+b_k)+$$

$$\psi^{(0)}(u-b_1+b_4+1)+\psi^{(0)}(u-b_2+b_4+1)+\psi^{(0)}(u-b_3+b_4+1)+\psi^{(0)}(u+1)-\log(z)\Bigg)-$$

$$\sum_{k=n+1}^{p} \psi^{(2)}(-u + a_k - b_4) - \sum_{k=1}^{n} \psi^{(2)}(u - a_k + b_4 + 1) + \sum_{k=m+1}^{q} \psi^{(2)}(u + b_4 - b_k + 1) + \sum_{k=5}^{m} \psi^{(2)}(-u - b_4 + b_k) +$$

$$\left.\psi^{(2)}(u - b_1 + b_4 + 1) + \psi^{(2)}(u - b_2 + b_4 + 1) + \psi^{(2)}(u - b_3 + b_4 + 1) + \psi^{(2)}(u + 1)\right) /;$$

$b_4 - b_3 \in \mathbb{N} \wedge b_3 - b_2 \in \mathbb{N} \wedge b_2 - b_1 \in \mathbb{N} \wedge u \in \mathbb{N} \wedge b_j - b_1 \notin \mathbb{Z} \wedge$

$5 \leq j \leq m \wedge$

$a_j - b_4 - 1 \notin \mathbb{N} \wedge 1 \leq j \leq n \wedge$

$-a_j + b_4 + u \notin \mathbb{N} \wedge n + 1 \leq j \leq p \wedge$

$b_j - b_4 - u - 1 \notin \mathbb{N} \wedge m + 1 \leq j \leq q$

```
Assuming[b_2 = b_1 + h_1, Assuming[b_3 = b_1 + h_1 + h_2, Assuming[b_4 = b_1 + h_1 + h_2 + h_3,
   (Table[{Residue[((∏_{k=1}^{n} Gamma[1-s-a_k]) ∏_{k=1}^{m} Gamma[s+b_k]) / ((∏_{k=n+1}^{p} Gamma[s+a_k]) ∏_{k=m+1}^{q} Gamma[1-s-b_k]) z^(-s), {s, -b_4 - u}, Assumptions→ {u ∈ Integers ∧
            u ≥ 0 ∧ h_1 ∈ Integers ∧ h_1 ≥ 0 ∧ h_2 ∈ Integers ∧ h_2 ≥ 0 ∧ h_3 ∈ Integers ∧ h_3 ≥ 0}] /
        (((-1)^(-b_1+b_2-b_3+b_4) (∏_{k=5}^{m} Gamma[-u-b_4+b_k]) ∏_{k=1}^{n} Gamma[1+u-a_k+b_4]) / (6 u! (u-b_1+b_4)! (u-b_2+b_4)! (u-b_3+b_4)! (∏_{k=n+1}^{p} Gamma[-u+a_k-b_4]) ∏_{k=m+1}^{q} Gamma[1+u+b_4-b_k]) z^(u+b_4)
          (PolyGamma[2, 1 + u] + PolyGamma[2, 1 + u - b_1 + b_4] +
            PolyGamma[2, 1 + u - b_2 + b_4] + PolyGamma[2, 1 + u - b_3 + b_4] +
            (-Log[z] + PolyGamma[0, 1 + u] + PolyGamma[0, 1 + u - b_1 + b_4] +
                PolyGamma[0, 1 + u - b_2 + b_4] + PolyGamma[0, 1 + u - b_3 + b_4] -
                ∑_{k=n+1}^{p} PolyGamma[0, -u + a_k - b_4] - ∑_{k=1}^{n} PolyGamma[0, 1 + u - a_k + b_4] +
                ∑_{k=m+1}^{q} PolyGamma[0, 1 + u + b_4 - b_k] + ∑_{k=5}^{m} PolyGamma[0, -u - b_4 + b_k])^3 +
            (-Log[z] + PolyGamma[0, 1 + u] + PolyGamma[0, 1 + u - b_1 + b_4] +
                PolyGamma[0, 1 + u - b_2 + b_4] + PolyGamma[0, 1 + u - b_3 + b_4] -
                ∑_{k=n+1}^{p} PolyGamma[0, -u + a_k - b_4] - ∑_{k=1}^{n} PolyGamma[0, 1 + u - a_k + b_4] +
                ∑_{k=m+1}^{q} PolyGamma[0, 1 + u + b_4 - b_k] + ∑_{k=5}^{m} PolyGamma[0, -u - b_4 + b_k])
             (4 π^2 + 3 (-PolyGamma[1, 1 + u] - PolyGamma[1, 1 + u - b_1 + b_4] -
                    PolyGamma[1, 1 + u - b_2 + b_4] - PolyGamma[1, 1 + u - b_3 + b_4] -
                    ∑_{k=n+1}^{p} PolyGamma[1, -u + a_k - b_4] +
                    ∑_{k=1}^{n} PolyGamma[1, 1 + u - a_k + b_4] - ∑_{k=m+1}^{q} PolyGamma[1,
                        1 + u + b_4 - b_k] + ∑_{k=5}^{m} PolyGamma[1, -u - b_4 + b_k])) -
            ∑_{k=n+1}^{p} PolyGamma[2, -u + a_k - b_4] - ∑_{k=1}^{n} PolyGamma[2, 1 + u - a_k + b_4] +
            ∑_{k=m+1}^{q} PolyGamma[2, 1 + u + b_4 - b_k] + ∑_{k=5}^{m} PolyGamma[2, -u - b_4 + b_k]))},
        {m, 4, 6}, {n, 0, 2}, {p, n, 5}, {q, m, 6}] // Simplify[
         #,
         Assumptions→ {k ∈ Integers}] &) //
      Simplify[#, Assumptions→ {u ∈ Integers ∧ u ≥ 0 ∧ h_1 ∈ Integers ∧ h_1 ≥ 0 ∧
              h_2 ∈ Integers ∧ h_2 ≥ 0 ∧ h_3 ∈ Integers ∧ h_3 ≥ 0}] &]]] //
  Simplify[#, Assumptions→ {u ∈ Integers ∧
        u ≥
          0 ∧ h_1 ∈
          Integers ∧ h_1 ≥
          0 ∧ h_2 ∈
          Integers ∧ h_2 ≥
          0 ∧ h_3 ∈
          Integers ∧ h_3 ≥
          0}] &
```

```
Assuming[b2 = b1 + h1, Assuming[b3 = b1 + h1 + h2, Assuming[b4 = b1 + h1 + h2 + h3,
   (Table[{Residue[(Product[Gamma[1-s-ak], {k,1,n}] Product[Gamma[s+bk], {k,1,m}]) /
         ((Product[Gamma[s+ak], {k,n+1,p}]) Product[Gamma[1-s-bk], {k,m+1,q}]) z^-s, {s, -b4 - u}, Assumptions→ {u ∈ Integers∧
               u ≥ 0 ∧ h1 ∈ Integers∧ h1 ≥ 0 ∧ h2 ∈ Integers∧ h2 ≥ 0 ∧ h3 ∈ Integers∧ h3 ≥ 0}] /
          (GammaResidueLeft[{Table[bi, {i, m}], Table[1 - ai, {i, n}]},
             {Table[ai, {i, n + 1, p}], Table[1 - bi, {i, m + 1, q}]}, {b4, 4, u}, z])}, {m, 4, 6},
       {n, 0, 2}, {p, n, 5}, {q, m, 6}] // Simplify[#, Assumptions→ {k ∈ Integers}] &) //
     Simplify[#, Assumptions→ {u ∈ Integers∧ u ≥ 0 ∧ h1 ∈ Integers∧ h1 ≥ 0 ∧
          h2 ∈ Integers∧ h2 ≥ 0 ∧ h3 ∈ Integers∧ h3 ≥ 0}] &]]] //
  Simplify[#, Assumptions→ {u ∈ Integers∧ u ≥ 0 ∧ h1 ∈ Integers∧ h1 ≥ 0 ∧ h2 ∈ Integers∧
       h2 ≥ 0 ∧ h3 ∈ Integers∧ h3 ≥ 0}] &
```

Case of right quartic poles

$$\text{Residue}\left[\frac{\left(\prod_{k=1}^{m}\text{Gamma}[s+b_k]\right)\prod_{k=1}^{n}\text{Gamma}[1-s-a_k]}{\left(\prod_{k=n+1}^{p}\text{Gamma}[s+a_k]\right)\prod_{k=m+1}^{q}\text{Gamma}[1-s-b_k]}\,z^{-s},\ \{s,\,1-a_4+u\}\right] ==$$

$$\left(\left((-1)^{-1+a_1-a_2+a_3-a_4}\left(\prod_{k=1}^{m}\text{Gamma}[1+u-a_4+b_k]\right)\prod_{k=5}^{n}\text{Gamma}[-u+a_4-a_k]\right)\Big/\left(6\,u!\ (u+a_1-a_4)!\right.\right.$$
$$\left.\left.(u+a_2-a_4)!\ (u+a_3-a_4)!\left(\prod_{k=n+1}^{p}\text{Gamma}[1+u-a_4+a_k]\right)\prod_{k=m+1}^{q}\text{Gamma}[-u+a_4-b_k]\right)\right)$$

$$z^{-1-u+a_4}\Bigg(\text{PolyGamma}[2,1+u]+\text{PolyGamma}[2,1+u+a_1-a_4]+$$
$$\text{PolyGamma}[2,1+u+a_2-a_4]+\text{PolyGamma}[2,1+u+a_3-a_4]+$$
$$\Bigg(\text{Log}[z]+\text{PolyGamma}[0,1+u]+\text{PolyGamma}[0,1+u+a_1-a_4]+\text{PolyGamma}[0,1+u+a_2-a_4]+$$
$$\text{PolyGamma}[0,1+u+a_3-a_4]+\sum_{k=5}^{n}\text{PolyGamma}[0,-u+a_4-a_k]+\sum_{k=n+1}^{p}\text{PolyGamma}[0,$$
$$1+u-a_4+a_k]-\sum_{k=m+1}^{q}\text{PolyGamma}[0,-u+a_4-b_k]-\sum_{k=1}^{m}\text{PolyGamma}[0,1+u-a_4+b_k]\Bigg)^3+$$
$$\Bigg(\text{Log}[z]+\text{PolyGamma}[0,1+u]+\text{PolyGamma}[0,1+u+a_1-a_4]+\text{PolyGamma}[0,1+u+a_2-a_4]+$$
$$\text{PolyGamma}[0,1+u+a_3-a_4]+\sum_{k=5}^{n}\text{PolyGamma}[0,-u+a_4-a_k]+\sum_{k=n+1}^{p}\text{PolyGamma}[0,$$
$$1+u-a_4+a_k]-\sum_{k=m+1}^{q}\text{PolyGamma}[0,-u+a_4-b_k]-\sum_{k=1}^{m}\text{PolyGamma}[0,1+u-a_4+b_k]\Bigg)$$

$$\left(4\pi^2 + 3\left(-\text{PolyGamma}[1, 1+u] - \text{PolyGamma}[1, 1+u+a_1-a_4] - \text{PolyGamma}[1, 1+u+a_2-a_4] - \text{PolyGamma}[1, 1+u+a_3-a_4] + \sum_{k=5}^{n}\text{PolyGamma}[1, -u+a_4-a_k] - \sum_{k=n+1}^{p}\text{PolyGamma}[1, 1+u-a_4+a_k] - \sum_{k=m+1}^{q}\text{PolyGamma}[1, -u+a_4-b_k] + \sum_{k=1}^{m}\text{PolyGamma}[1, 1+u-a_4+b_k]\right)\right) + \sum_{k=5}^{n}\text{PolyGamma}[2, -u+a_4-a_k] + \sum_{k=n+1}^{p}\text{PolyGamma}[2, 1+u-a_4+a_k] - \sum_{k=m+1}^{q}\text{PolyGamma}[2, -u+a_4-b_k] - \sum_{k=1}^{m}\text{PolyGamma}[2, 1+u-a_4+b_k]\Big) \;/;$$

$a_3 - a_4 \in \text{Integers} \&\& a_3 - a_4 \geq 0 \&\&$
$a_2 -$
$a_3 \in \text{Integers} \&\&$
$a_2 - a_3 \geq 0 \&\& a_1 - a_2 \in \text{Integers} \&\&$
$a_1 - a_2 \geq$
$0 \&\& u \in$
$\text{Integers} \&\& u \geq$
$0 \&\& a_1 - a_j \notin$
$\text{Integers} \&\& 5 \leq$
$j \leq$
$n \&\&$
$!\,(a_4 - b_j \in \text{Integers} \&\& -1 + a_4 - b_j \geq 0) \&\&$
$1 \leq$
$j \leq$
$m \&\&$
$!\,(u - a_4 + b_j \in \text{Integers} \&\& u - a_4 + b_j \geq 0) \&\&$
$m + 1 \leq$
$j \leq$
$q \&\&$
$!\,(-u + a_4 - a_j \in \text{Integers} \&\& -1 - u + a_4 - a_j \geq 0) \&\&$
$n + 1 \leq j \leq p$

$$\text{res}_s\left(\frac{\left(\prod_{k=1}^{m}\Gamma(s+b_k)\right)\prod_{k=1}^{n}\Gamma(-s-a_k+1)}{\left(\prod_{k=n+1}^{p}\Gamma(s+a_k)\right)\prod_{k=m+1}^{q}\Gamma(-s-b_k+1)}\, z^{-s}\right)(1-a_4+u) = \frac{(-1)^{a_1-a_2+a_3-a_4-1}\left(\prod_{k=1}^{m}\Gamma(u-a_4+b_k+1)\right)\prod_{k=5}^{n}\Gamma(-u+a_4-a_k)}{6\,u!\,(a_1-a_4+u)!\,(a_2-a_4+u)!\,(a_3-a_4+u)!\left(\prod_{k=n+1}^{p}\Gamma(u-a_4+a_k+1)\right)\prod_{k=m+1}^{q}\Gamma(-u+a_4-b_k)}\, z^{a_4-u-1}$$

$$\Bigg(\Bigg(-\sum_{k=m+1}^{q}\psi^{(0)}(-u+a_4-b_k)-\sum_{k=1}^{m}\psi^{(0)}(u-a_4+b_k+1)+\sum_{k=n+1}^{p}\psi^{(0)}(u-a_4+a_k+1)+\sum_{k=5}^{n}\psi^{(0)}(-u+a_4-a_k)+$$
$$\psi^{(0)}(u+a_1-a_4+1)+\psi^{(0)}(u+a_2-a_4+1)+\psi^{(0)}(u+a_3-a_4+1)+\psi^{(0)}(u+1)+\log(z)\Bigg)^3+$$
$$\Bigg(3\Bigg(-\sum_{k=m+1}^{q}\psi^{(1)}(-u+a_4-b_k)+\sum_{k=1}^{m}\psi^{(1)}(u-a_4+b_k+1)-\sum_{k=n+1}^{p}\psi^{(1)}(u-a_4+a_k+1)+\sum_{k=5}^{n}\psi^{(1)}(-u+a_4-a_k)-$$
$$\psi^{(1)}(u+a_1-a_4+1)-\psi^{(1)}(u+a_2-a_4+1)-\psi^{(1)}(u+a_3-a_4+1)-\psi^{(1)}(u+1)\Bigg)+4\,\pi^2\Bigg)$$
$$\Bigg(-\sum_{k=m+1}^{q}\psi^{(0)}(-u+a_4-b_k)-\sum_{k=1}^{m}\psi^{(0)}(u-a_4+b_k+1)+\sum_{k=n+1}^{p}\psi^{(0)}(u-a_4+a_k+1)+\sum_{k=5}^{n}\psi^{(0)}(-u+a_4-a_k)+$$
$$\psi^{(0)}(u+a_1-a_4+1)+\psi^{(0)}(u+a_2-a_4+1)+\psi^{(0)}(u+a_3-a_4+1)+\psi^{(0)}(u+1)+\log(z)\Bigg)-$$
$$\sum_{k=m+1}^{q}\psi^{(2)}(-u+a_4-b_k)-\sum_{k=1}^{m}\psi^{(2)}(u-a_4+b_k+1)+\sum_{k=n+1}^{p}\psi^{(2)}(u-a_4+a_k+1)+\sum_{k=5}^{n}\psi^{(2)}(-u+a_4-a_k)+$$
$$\psi^{(2)}(u+a_1-a_4+1)+\psi^{(2)}(u+a_2-a_4+1)+\psi^{(2)}(u+a_3-a_4+1)+\psi^{(2)}(u+1)\Bigg)\,/;$$

$a_3-a_4\in\mathbb{N}\land a_2-a_3\in\mathbb{N}\land a_1-a_2\in\mathbb{N}\land u\in\mathbb{N}\land a_1-a_j\notin\mathbb{Z}\land$
$5\le j\le n\land$
$a_4-b_j-1\notin\mathbb{N}\land 1\le j\le m\land$
$-a_4+b_j+u\notin\mathbb{N}\land m+1\le j\le q\land$
$-a_j+a_4-u-1\notin\mathbb{N}\land n+1\le j\le p$

```
Assuming[a2 = a1 - h1, Assuming[a3 = a1 - h1 - h2, Assuming[a4 = a1 - h1 - h2 - h3,
    (Table[{Residue[((∏_{k=1}^m Gamma[s+bk]) ∏_{k=1}^n Gamma[1-s-ak]) / ((∏_{k=n+1}^p Gamma[s+ak]) ∏_{k=m+1}^q Gamma[1-s-bk]) z^-s, {s, 1 - a4 + u}, Assumptions→ {u ∈ Integers∧
              u ≥ 0 ∧ h1 ∈ Integers∧ h1 ≥ 0 ∧ h2 ∈ Integers∧ h2 ≥ 0 ∧ h3 ∈ Integers∧ h3 ≥ 0}] /
          (((-1)^(-1+a1-a2+a3-a4) (∏_{k=1}^m Gamma[1+u-a4+bk]) ∏_{k=5}^n Gamma[-u+a4-ak]) /
            (6 u! (u+a1-a4)! (u+a2-a4)! (u+a3-a4)! (∏_{k=n+1}^p Gamma[1+u-a4+ak]) ∏_{k=m+1}^q Gamma[-u+a4-bk])
          z^(-1-u+a4) (PolyGamma[2, 1 + u] + PolyGamma[2, 1 + u + a1 - a4] +
              PolyGamma[2, 1 + u + a2 - a4] + PolyGamma[2, 1 + u + a3 - a4] +
              (Log[z] + PolyGamma[0, 1 + u] + PolyGamma[0, 1 + u + a1 - a4] +
                  PolyGamma[0, 1 + u + a2 - a4] + PolyGamma[0, 1 + u + a3 - a4] +
                  ∑_{k=5}^n PolyGamma[0, -u + a4 - ak] + ∑_{k=n+1}^p PolyGamma[0, 1 + u - a4 + ak] -
                  ∑_{k=m+1}^q PolyGamma[0, -u + a4 - bk] - ∑_{k=1}^m PolyGamma[0, 1 + u - a4 + bk])^3 +
              (Log[z] + PolyGamma[0, 1 + u] + PolyGamma[0, 1 + u + a1 - a4] +
                  PolyGamma[0, 1 + u + a2 - a4] + PolyGamma[0, 1 + u + a3 - a4] +
                  ∑_{k=5}^n PolyGamma[0, -u + a4 - ak] + ∑_{k=n+1}^p PolyGamma[0, 1 + u - a4 + ak] -
                  ∑_{k=m+1}^q PolyGamma[0, -u + a4 - bk] - ∑_{k=1}^m PolyGamma[0, 1 + u - a4 + bk])
                (4 π^2 + 3 (-PolyGamma[1, 1 + u] - PolyGamma[1, 1 + u + a1 - a4] -
                      PolyGamma[1, 1 + u + a2 - a4] - PolyGamma[1, 1 + u + a3 - a4] +
                      ∑_{k=5}^n PolyGamma[1, -u + a4 - ak] -
                      ∑_{k=n+1}^p PolyGamma[1, 1 + u - a4 + ak] - ∑_{k=m+1}^q PolyGamma[1,
                          -u + a4 - bk] + ∑_{k=1}^m PolyGamma[1, 1 + u - a4 + bk])) +
              ∑_{k=5}^n PolyGamma[2, -u + a4 - ak] + ∑_{k=n+1}^p PolyGamma[2, 1 + u - a4 + ak] -
              ∑_{k=m+1}^q PolyGamma[2, -u + a4 - bk] - ∑_{k=1}^m PolyGamma[2, 1 + u - a4 + bk]))},
        {m, 0, 2}, {n, 4, 5}, {p, n, 6}, {q, m, 6}] // Simplify[
          #,
          Assumptions→ {k ∈ Integers}] &) //
      Simplify[#, Assumptions→ {u ∈ Integers∧ u ≥ 0 ∧ h1 ∈ Integers∧ h1 ≥ 0 ∧
              h2 ∈ Integers∧ h2 ≥ 0 ∧ h3 ∈ Integers∧ h3 ≥ 0}] &]]] //
  Simplify[#, Assumptions→ {u ∈ Integers∧
          u ≥
            0 ∧ h1 ∈
            Integers∧ h1 ≥
            0 ∧ h2 ∈
            Integers∧ h2 ≥
            0 ∧ h3 ∈
            Integers∧ h3 ≥
            0}] &
```

```
Assuming[a2 = a1 - h1, Assuming[a3 = a1 - h1 - h2, Assuming[a4 = a1 - h1 - h2 - h3,
   (Table[{Residue[((∏_{k=1}^m Gamma[s+bk]) ∏_{k=1}^n Gamma[1-s-ak]) / ((∏_{k=n+1}^p Gamma[s+ak]) ∏_{k=m+1}^q Gamma[1-s-bk]) z^-s, {s, 1 - a4 + u}, Assumptions→ {u ∈ Integers∧
            u ≥ 0 ∧ h1 ∈ Integers∧ h1 ≥ 0 ∧ h2 ∈ Integers∧ h2 ≥ 0 ∧ h3 ∈ Integers∧ h3 ≥ 0}] /
         (GammaResidueRight[{Table[bi, {i, m}], Table[1 - ai, {i, n}]},
            {Table[ai, {i, n + 1, p}], Table[1 - bi, {i, m + 1, q}]}, {1 - a4, 4, u}, z])}, {m, 0, 2},
       {n, 4, 5}, {p, n, 6}, {q, m, 6}] // Simplify[#, Assumptions→ {k ∈ Integers}] &) //
     Simplify[#, Assumptions→ {u ∈ Integers∧ u ≥ 0 ∧ h1 ∈ Integers∧ h1 ≥ 0 ∧
         h2 ∈ Integers∧ h2 ≥ 0 ∧ h3 ∈ Integers∧ h3 ≥ 0}] &]]] //
  Simplify[#, Assumptions→ {u ∈ Integers∧ u ≥ 0 ∧ h1 ∈ Integers∧ h1 ≥ 0 ∧ h2 ∈ Integers∧
        h2 ≥ 0 ∧ h3 ∈ Integers∧ h3 ≥ 0}] &
Residue[((∏_{k=1}^m Gamma[s + bk]) ∏_{k=1}^n Gamma[1 - s - ak]) / ((∏_{k=n+1}^p Gamma[s + ak]) ∏_{k=m+1}^q Gamma[1 - s - bk]) z^-s, {s, 1 - a4 + u}] ==
   GammaResidueRight[{Table[bi, {i, m}], Table[1 - ai, {i, n}]},
    {Table[ai, {i, n + 1, p}], Table[1 - bi, {i, m + 1, q}]}, {1 - a4, 4, u}, z] /;
  a3 - a4 ∈ Integers&& a3 - a4 ≥ 0 && a2 - a3 ∈ Integers&& a2 - a3 ≥ 0 && a1 - a2 ∈ Integers&&
   a1 - a2 ≥ 0 && u ∈ Integers&& u ≥ 0 && a1 - aj ∉ Integers&& 5 ≤ j ≤ n &&
   ! (a4 - bj ∈ Integers&& -1 + a4 - bj ≥ 0) && 1 ≤ j ≤ m && ! (u - a4 + bj ∈ Integers&& u - a4 + bj ≥ 0) &&
   m + 1 ≤ j ≤ q && ! (-u + a4 - aj ∈ Integers&& -1 - u + a4 - aj ≥ 0) && n + 1 ≤ j ≤ p
```

$$\operatorname{res}_s\left(\frac{\left(\prod_{k=1}^{m}\Gamma(s+b_k)\right)\prod_{k=1}^{n}\Gamma(-s-a_k+1)}{\left(\prod_{k=n+1}^{p}\Gamma(s+a_k)\right)\prod_{k=m+1}^{q}\Gamma(-s-b_k+1)}\,z^{-s}\right)(1-a_4+u) == \text{GammaResidueRight}(\{\text{Table}[b_i,\{i,m\}],\text{Table}[1-a_i,\{i,n\}]\},\{\text{Table}[a_i,\{i,n+1,p\}],\text{Table}[1-b_i,\{i,m+1,q\}]\},\{1-a_4,4,u\},z)\ /;$$

$$a_3-a_4\in\mathbb{N}\wedge a_2-a_3\in\mathbb{N}\wedge a_1-a_2\in\mathbb{N}\wedge u\in\mathbb{N}\wedge a_1-a_j\notin\mathbb{Z}\wedge 5\le j\le n\wedge a_4-b_j-1\notin\mathbb{N}\wedge 1\le j\le m\wedge -a_4+b_j+u\notin\mathbb{N}\wedge m+1\le j\le q\wedge -a_j+a_4-u-1\notin\mathbb{N}\wedge n+1\le j\le p$$

```
Assuming[a2 = a1 - h1, Assuming[a3 = a1 - h1 - h2, Assuming[a4 = a1 - h1 - h2 - h3,
   (Table[{Residue[((∏_{k=1}^m Gamma[s+bk]) ∏_{k=1}^n Gamma[1-s-ak]) / ((∏_{k=n+1}^p Gamma[s+ak]) ∏_{k=m+1}^q Gamma[1-s-bk]) z^-s, {s, 1 - a4 + u}, Assumptions→ {u ∈ Integers∧
              u ≥ 0 ∧ h1 ∈ Integers∧ h1 ≥ 0 ∧ h2 ∈ Integers∧ h2 ≥ 0 ∧ h3 ∈ Integers∧ h3 ≥ 0}] /
         (((-1)^(-1+a1-a2+a3-a4) (∏_{k=1}^m Gamma[1+u-a4+bk]) ∏_{k=5}^n Gamma[-u+a4-ak]) / (6 u! (u+a1-a4)! (u+a2-a4)! (u+a3-a4)! (∏_{k=n+1}^p Gamma[1+u-a4+ak]) ∏_{k=m+1}^q Gamma[-u+a4-bk])
           z^(-1-u+a4) (PolyGamma[2, 1 + u] + PolyGamma[2, 1 + u + a1 - a4] +
              PolyGamma[2, 1 + u + a2 - a4] + PolyGamma[2, 1 + u + a3 - a4] +
              (Log[z] + PolyGamma[0, 1 + u] + PolyGamma[0, 1 + u + a1 - a4] +
                  PolyGamma[0, 1 + u + a2 - a4] + PolyGamma[0, 1 + u + a3 - a4] +
                  ∑_{k=5}^n PolyGamma[0, -u + a4 - ak] + ∑_{k=n+1}^p PolyGamma[0, 1 + u - a4 + ak] -
                  ∑_{k=m+1}^q PolyGamma[0, -u + a4 - bk] - ∑_{k=1}^m PolyGamma[0, 1 + u - a4 + bk])^3 +
              (Log[z] + PolyGamma[0, 1 + u] + PolyGamma[0, 1 + u + a1 - a4] +
                  PolyGamma[0, 1 + u + a2 - a4] + PolyGamma[0, 1 + u + a3 - a4] +
                  ∑_{k=5}^n PolyGamma[0, -u + a4 - ak] + ∑_{k=n+1}^p PolyGamma[0, 1 + u - a4 + ak] -
                  ∑_{k=m+1}^q PolyGamma[0, -u + a4 - bk] - ∑_{k=1}^m PolyGamma[0, 1 + u - a4 + bk])
               (4 π^2 + 3 (-PolyGamma[1, 1 + u] - PolyGamma[1, 1 + u + a1 - a4] -
                      PolyGamma[1, 1 + u + a2 - a4] - PolyGamma[1, 1 + u + a3 - a4] +
                      ∑_{k=5}^n PolyGamma[1, -u + a4 - ak] -
                      ∑_{k=n+1}^p PolyGamma[1, 1 + u - a4 + ak] - ∑_{k=m+1}^q PolyGamma[1,
                          -u + a4 - bk] + ∑_{k=1}^m PolyGamma[1, 1 + u - a4 + bk])) +
              ∑_{k=5}^n PolyGamma[2, -u + a4 - ak] + ∑_{k=n+1}^p PolyGamma[2, 1 + u - a4 + ak] -
              ∑_{k=m+1}^q PolyGamma[2, -u + a4 - bk] - ∑_{k=1}^m PolyGamma[2, 1 + u - a4 + bk]))},
        {m, 0, 2}, {n, 4, 5}, {p, n, 6}, {q, m, 6}] // Simplify[
         #,
         Assumptions→ {k ∈ Integers}] &) //
     Simplify[#, Assumptions→ {u ∈ Integers∧ u ≥ 0 ∧ h1 ∈ Integers∧ h1 ≥ 0 ∧
            h2 ∈ Integers∧ h2 ≥ 0 ∧ h3 ∈ Integers∧ h3 ≥ 0}] &]]] //
  Simplify[#, Assumptions→ {u ∈ Integers∧
          u ≥
           0 ∧ h1 ∈
           Integers∧ h1 ≥
           0 ∧ h2 ∈
           Integers∧ h2 ≥
           0 ∧ h3 ∈
           Integers∧ h3 ≥
           0}] &
```

$$\text{Assuming}\Big[a_2 = a_1 - h_1,\ \text{Assuming}\Big[a_3 = a_1 - h_1 - h_2,\ \text{Assuming}\Big[a_4 = a_1 - h_1 - h_2 - h_3,$$

$$\Big(\text{Table}\Big[\Big\{\text{Residue}\Big[\frac{\left(\prod_{k=1}^{m}\text{Gamma}[s+b_k]\right)\prod_{k=1}^{n}\text{Gamma}[1-s-a_k]}{\left(\prod_{k=n+1}^{p}\text{Gamma}[s+a_k]\right)\prod_{k=m+1}^{q}\text{Gamma}[1-s-b_k]}\, z^{-s},\ \{s, 1-a_4+u\},\ \text{Assumptions}\to\{u\in\text{Integers}\wedge$$

$$u\geq 0\wedge h_1\in\text{Integers}\wedge h_1\geq 0\wedge h_2\in\text{Integers}\wedge h_2\geq 0\wedge h_3\in\text{Integers}\wedge h_3\geq 0\}\Big]\Big/$$

$$(\text{GammaResidueRight}[\{\text{Table}[b_i, \{i, m\}], \text{Table}[1-a_i, \{i, n\}]\},$$

$$\{\text{Table}[a_i, \{i, n+1, p\}], \text{Table}[1-b_i, \{i, m+1, q\}]\}, \{1-a_4, 4, u\}, z])\Big\}, \{m, 0, 2\},$$

$$\{n, 4, 5\}, \{p, n, 6\}, \{q, m, 6\}\Big]\ //\ \text{Simplify}[\#, \text{Assumptions}\to\{k\in\text{Integers}\}]\ \&\Big)\ //$$

$$\text{Simplify}[\#, \text{Assumptions}\to\{u\in\text{Integers}\wedge u\geq 0\wedge h_1\in\text{Integers}\wedge h_1\geq 0\wedge$$

$$h_2\in\text{Integers}\wedge h_2\geq 0\wedge h_3\in\text{Integers}\wedge h_3\geq 0\}]\ \&\Big]\Big]\Big]\ //$$

$$\text{Simplify}[\#, \text{Assumptions}\to\{u\in\text{Integers}\wedge u\geq 0\wedge h_1\in\text{Integers}\wedge h_1\geq 0\wedge h_2\in\text{Integers}\wedge$$

$$h_2\geq 0\wedge h_3\in\text{Integers}\wedge h_3\geq 0\}]\ \&$$

Case of left u-th order poles

$$\text{Residue}\Big[\frac{\left(\prod_{j=1}^{m}\text{Gamma}[b_j+s]\right)\prod_{i=1}^{n}\text{Gamma}[1-a_i-s]}{\left(\prod_{i=n+1}^{p}\text{Gamma}[a_i+s]\right)\prod_{j=m+1}^{q}\text{Gamma}[1-b_j-s]}\, z^{-s},\ \{s, -b_m-\grave{\imath}_m\}\Big] ==$$

$$\frac{z^{b_m+\grave{\imath}_m}\,(-1)^{\sum_{j=1}^{u}\grave{\imath}_{m-j+1}}}{(u-1)!}\left(\sum_{k=0}^{u-1}\text{Binomial}[u-1, k]\right.$$

$$\left(\frac{\text{KroneckerDelta}[u-1-k]}{\left(\prod_{i=n+1}^{p}\text{Gamma}[a_i-\grave{\imath}_m-b_m]\right)\prod_{j=m-u+1}^{q}\text{Gamma}[1-b_j+\grave{\imath}_m+b_m]}+\text{UnitStep}[u-k-2]\right.$$

$$\text{BellY}\Big[\text{Table}\Big[\Big\{(-1)^j\, j!\,\left(\left(\prod_{i=n+1}^{p}\text{Gamma}[a_i-\grave{\imath}_m-b_m]\right)\prod_{j=m-u+1}^{q}\text{Gamma}[1-b_j+\grave{\imath}_m+b_m\right)^{-1-j},$$

$$\sum_{i=0}^{j}\text{Binomial}[j, i]\left(\sum_{k_{m-u+1}=0}^{j-i}\ldots\sum_{k_q=0}^{j-i}\text{KroneckerDelta}\Big[j-i, \sum_{j=m-u+1}^{q}k_j\Big]\right.$$

$$\text{Multinomial}[k_{m-u+1}, \ldots, k_q]\prod_{j=m-u+1}^{q}\Big(\text{Gamma}[1-b_j+\grave{\imath}_m+b_m]$$

$$(-1)^{k_j}\Big(\text{KroneckerDelta}[k_j]+\text{UnitStep}[k_j-1]\,\text{BellY}\Big[$$

$$\left.\text{Table}[\{1, \text{PolyGamma}[-1+t, 1-b_j+\grave{\imath}_m+b_m]\}, \{t, k_j\}]]))\right)$$

$$\left(\sum_{k_{n+1}=0}^{i}\ldots\sum_{k_p=0}^{i}\text{KroneckerDelta}\Big[i, \sum_{j=n+1}^{p}k_j\Big]\,\text{Multinomial}[k_{n+1}, \ldots, k_p]\right.$$

$$\prod_{j=n+1}^{p}\Big(\text{Gamma}[a_j-\grave{\imath}_m-b_m]\,\Big(\text{KroneckerDelta}[k_j]+\text{UnitStep}[k_j-1]\,\text{BellY}\Big[$$

$$\text{Table}\left[\{1, \text{PolyGamma}[-1+t, a_j - ì_m - b_m]\}, \{t, k_j\}\right]))\Big)\Big\},$$

$$\{j, u-1-k\}]]\Big)\sum_{i=0}^{k}\sum_{j=0}^{k}\Big(\text{Multinomial}[i, j, k-i-j]\ \text{Piecewise}\Big[\{\{1, m === u\}\},$$

$$\Big(\sum_{k_1=0}^{i}\cdots\sum_{k_{m-u}=0}^{i}\text{KroneckerDelta}\Big[i, \sum_{j=1}^{m-u}k_j\Big]\ \text{Multinomial}[k_1, \ldots, k_{m-u}]$$

$$\prod_{j=1}^{m-u}(\text{Gamma}[b_j - ì_m - b_m]\ (\text{KroneckerDelta}[k_j] + \text{UnitStep}[k_j - 1]$$

$$\text{BellY}[\text{Table}[\{1, \text{PolyGamma}[t-1, b_j - ì_m - b_m]\}, \{t, k_j\}]]))\Big)\Big]$$

$$\Big(\sum_{k_1=0}^{j}\cdots\sum_{k_n=0}^{j}\text{KroneckerDelta}\Big[j, \sum_{j=1}^{n}k_j\Big]\ \text{Multinomial}[k_1, \ldots, k_n]\prod_{j=1}^{n}(\text{Gamma}[$$

$$1 - a_j + ì_m + b_m]\ (-1)^{k_j}\ (\text{KroneckerDelta}[k_j] + \text{UnitStep}[k_j - 1]$$

$$\text{BellY}[\text{Table}[\{1, \text{PolyGamma}[t-1, 1 - a_j + ì_m + b_m]\}, \{t, k_j\}]]))\Big)$$

$$(k-i-j)!\sum_{r=0}^{\text{Floor}\left[\frac{k-i-j}{2}\right]}\frac{(-\text{Log}[z])^{k-i-j-2r}\pi^{2r}}{(k-i-j-2r)!}\Big(\delta_r + \frac{\text{UnitStep}[r-1]}{r!}\ \text{BellY}\Big[$$

$$\text{Table}\Big[\Big\{(-1)^i\ \text{Pochhammer}[-u, i],$$

$$\frac{(-1)^{i-1}\,2\,(2^{2i-1}-1)\,i!\ \text{BernoulliB}[2i]}{(2i)!}\Big\}, \{i, r\}]]\Big)\Big)\Big)\ /;$$

$u \in \text{Integers} \&\&\ ì_{m-j+1} \in \text{Integers} \&\&\ ì_{m-j+1} \geq 0 \&\&\ 1 \leq j \leq$

$u \leq$

$m \&\&$

$b_{m-j+1} ==$

$b_m +$

$ì_m -$

$ì_{m-j+1} \&\&$

$0 \leq$

$j \leq$

$u \leq$

$m \&\&$

Not[

$a_i -$

$b_m -$
$\grave{\imath}_m \in$ Integers&& $a_i -$
$b_m -$
$\grave{\imath}_m \leq 0$ && $n +$
$1 \leq i \leq p$] &&
Not[$1 - b_j + b_m + \grave{\imath}_m \in$ Integers&& $1 - b_j + b_m + \grave{\imath}_m \leq$
0 &&
$m + 1 \leq j \leq q$]

$$\operatorname{res}\left(\frac{\left(\prod_{j=1}^{m}\Gamma(b_j+s)\right)\prod_{i=1}^{n}\Gamma(1-a_i-s)}{\left(\prod_{i=n+1}^{p}\Gamma(a_i+s)\right)\prod_{j=m+1}^{q}\Gamma(1-b_j-s)}\,z^{-s},\ \{s,\,-b_m-\grave{\imath}_m\}\right)=$$

$$\frac{z^{b_m+\grave{\imath}_m}\,(-1)^{\sum_{j=1}^{u}\grave{\imath}_{m-j+1}}\,\pi^u}{(u-1)!}\sum_{k=0}^{u-1}\binom{u-1}{k}\left(\frac{\delta_{u-k-1}}{\left(\prod_{i=n+1}^{p}\Gamma(a_i-b_m-\grave{\imath}_m)\right)\prod_{j=m-u+1}^{q}\Gamma(1-b_j+b_m+\grave{\imath}_m)}+\right.$$

$$\theta(u-k-2)\,\text{BellY}\Big[\text{Table}\Big[\Big\{(-1)^j\,j!\left(\left(\prod_{i=n+1}^{p}\Gamma(a_i-b_m-\grave{\imath}_m)\right)\prod_{j=m-u+1}^{q}\Gamma(1-b_j+b_m+\grave{\imath}_m)\right)^{-j-1},$$

$$\sum_{i=0}^{j}\binom{j}{i}\left(\sum_{k_{m-u+1}=0}^{j-i}\cdots\sum_{k_q=0}^{j-i}\delta_{j-i,\sum_{j=m-u+1}^{q}k_j}\,(k_{m-u+1}+\ldots+k_q;\,k_{m-u+1},\,\ldots,\,k_q)\prod_{j=m-u+1}^{q}\Gamma(1-b_j+b_m+\grave{\imath}_m)\right.$$

$$\left.(-\beta_j)^{k_j}\left(\delta_{k_j}+\theta(k_j-1)\,\text{BellY}\big[\text{Table}\big[\{1,\,\psi^{(t-1)}(1-b_j+b_m+\grave{\imath}_m)\},\,\{t,\,k_j\}\big]\big]\right)\right)$$

$$\sum_{k_{n+1}=0}^{i}\cdots\sum_{k_p=0}^{i}\delta_{i,\sum_{j=n+1}^{p}k_j}\,(k_{n+1}+\ldots+k_p;\,k_{n+1},\,\ldots,\,k_p)\prod_{j=n+1}^{p}\Gamma(a_i-b_m-\grave{\imath}_m)$$

$$\left.\left(\delta_{k_j}+\theta(k_j-1)\,\text{BellY}\big[\text{Table}\big[\{1,\,\psi^{(t-1)}(a_i-b_m-\grave{\imath}_m)\},\,\{t,\,k_j\}\big]\big]\right)\Big\},\,\{j,\,u-k-1\}\Big]\Big]\right)$$

$$\sum_{i=0}^{k}\sum_{j=0}^{k}(k;\,i,\,j,\,k-i-j)\left\{\begin{array}{l}1\\ \sum_{k_1=0}^{i}\cdots\sum_{k_{m-u}=0}^{i}\delta_{i,\sum_{j=1}^{m-u}k_j}\,(k_1+\ldots+k_{m-u};\,k_1,\,\ldots,\,k_{m-u})\\ \qquad\prod_{j=1}^{m-u}\Gamma(b_j-b_m-\grave{\imath}_m)\left(\delta_{k_j}+\ \theta(k_j-1)\,\text{BellY}\big[\text{Table}\big[\{1,\,\psi^{(t-1)}(b_j-b_m-\grave{\imath}_m)\},\,\{t,\,k_j\}\big]\big]\right)\end{array}\right.$$

$$\left.(-\alpha_j)^{k_j}\left(\delta_{k_j}+\theta(k_j-1)\,\text{BellY}\big[\text{Table}\big[\{1,\,\psi^{(t-1)}(1-a_j+b_m+\grave{\imath}_m)\},\,\{t,\,k_j\}\big]\big]\right)\right)(k-i-j)!$$

$$\sum_{r=0}^{\left\lfloor\frac{k-i-j}{2}\right\rfloor}\frac{(-\log(z))^{k-i-j-2r}\,\pi^{2r}}{(k-i-j-2r)!}\left(\delta_r+\frac{\theta(r-1)}{r!}\,\text{BellY}\Big[\text{Table}\Big[\Big\{(-1)^i\,(-u)_i,\,\frac{(-1)^{i-1}\,2\,(2^{2i-1}-1)\,i!\,B_{2i}}{(2i)!}\Big\},\,\{i,\,r\}\Big]\Big]\right)/;$$

$u \in \mathbb{N} \land \grave{\imath}_{m-j+1} \in \mathbb{N} \land 1 \leq j \leq u \leq$
$m \land$
$b_{m-j+1} =$
$b_m +$
$\grave{\imath}_m -$
$\grave{\imath}_{m-j+1} \land 0 \leq$
$j \leq$
$u \leq$

$$m \wedge$$
$$\neg (a_i - b_m - \imath_m \in \mathbb{Z} \wedge a_i - b_m - \imath_m \leq 0 \wedge n + 1 \leq i \leq p) \wedge$$
$$\neg \left(b_m + \imath_m - b_j + 1 \in \mathbb{Z} \wedge b_m + \imath_m - b_j + 1 \leq 0 \wedge m + 1 \leq j \leq q\right)$$

```
Clear[restrictedMultidimensionalSum]; restrictedMultidimensionalSum[body_, k_, {Q_, M_}] :=
  Module[{body1, kIterators},
    body1 = body /. Subscript[k, Q] → M - Sum[Subscript[k, j], {j, 1, Q - 1}];
   kIterators = Table[{Subscript[k, j], 0, M - Sum[Subscript[k, i], {i, 1, j - 1}]}, {j, Q - 1}];
   If[Q === 1, body /. Subscript[k, j_] :→ M,
     Sum[Evaluate[body1], Evaluate[Sequence @@ kIterators]]]]
```

```
Ans = Quiet[Simplify[Table[With[{m = 7, n = 1, p = 2, q = 8},
       {Solve[Table[Subscript[b, m - j + 1] + s == -Subscript[ì, m - j + 1] + ϵ, {j, u}],
           Union[{s}, Table[Subscript[b, m - j + 1], {j, 2, u}]]][[
    1]], Simplify[Residue1[(Product[Gamma[Subscript[b, j] + s], {j, 1, m}] *
                 Product[Gamma[1 - Subscript[a, i] - s], {i, 1, n}]) /
               (Product[Gamma[Subscript[a, i] + s], {i, n + 1, p}] *
     Product[Gamma[1 - Subscript[b, j] - s], {j, m + 1, q}]) / z^s,
           {ϵ, 0}, Assumptions → {And @@ Flatten[
                 Union[Table[{Element[Subscript[ì, m - j + 1], Integers]}, {j, 1, u}], Table[
      {Subscript[ì, m - j + 1] ≥ 0}, {j, 1, u}]]]}]] /.
          Solve[Table[Subscript[b, m - j + 1] + s == -Subscript[ì, m - j + 1] + ϵ, {j, u}],
    Union[{s}, Table[Subscript[b, m - j + 1], {j, 2, u}]]][[1]],
        z^(Subscript[b, m] + Subscript[ì, m]) *
          (-1)^Sum[Subscript[ì, m - j + 1], {j, 1, u}] * Pi^u *
    Module[{qqq, res0, res}, qqq[k_, u_] := (k! *
                 Sum[((-Log[z])^(k - 2*r) / (k - 2*r)!) * Pi^(2*r) * (KroneckerDelta[r] +
                     (UnitStep[r - 1] / r!) * BellY[Table[{(-1)^i * Pochhammer[-u, i],
      ((-1)^(i - 1) * 2 * (2^(2*i - 1) - 1) * i! * BernoulliB[2*i]) / (2*i)!}, {i, 1, r}]]),
                  {r, 0, Floor[k / 2]}]) / Pi^u;
     res0 = (1 / (u - 1)!) * Sum[Binomial[u - 1, k] *
                   (KroneckerDelta[u - 1 - k] / (Product[Gamma[Subscript[a, i] -
                          Subscript[b, m] - Subscript[ì, m]], {i, n + 1, p}] *
       Product[Gamma[1 - Subscript[b, j] + Subscript[b, m] + Subscript[ì, m]],
                       {j, m - u + 1, q}]) + UnitStep[u - k - 2] *
       BellY[Table[{(-1)^j * j! * (Product[Gamma[Subscript[a, i] - Subscript[b, m] -
                                 Subscript[ì, m]], {i, n + 1, p}] *
                                Product[Gamma[1 - Subscript[b, j] +
                                 Subscript[b, m] + Subscript[ì, m]],
        {j, m - u + 1, q}])^(-1 - j), Sum[Binomial[j, i] *
                             restrictedMultidimensionalSum[Multinomial @@
                                 Table[Subscript[k, j], {j, p - n}] * Product[
```

```
Gamma[Subscript[a, j] - Subscript[b, m] - Subscript[ì, m]] * (KroneckerDelta[Subscript[k,
        j - n]] + BellY[Table1[{1, PolyGamma[
        -1 + t, Subscript[a, j] - Subscript[b, m] -
  Subscript[ì, m]]}, {t, Subscript[k, j - n]}]]), {j, n + 1, p}], k, {p - n, i}] *
      restrictedMultidimensionalSum[Multinomial @@
        Table[Subscript[k, j], {j, q - m + u}] *
Product[Gamma[1 - Subscript[b, j] + Subscript[b, m] + Subscript[ì, m]] *
        (-1)^Subscript[k, j - m + u] * (KroneckerDelta[
        Subscript[k, j - m + u]] + BellY[Table1[
  {1, PolyGamma[-1 + t, 1 - Subscript[b, j] + Subscript[b, m] + Subscript[ì, m]]},
        {t, Subscript[k, j - m + u]}]]),
        {j, m - u + 1, q}], k, {q - m + u, j - i}],
{i, 0, j}]]} /. Table1 → Table /. BellY[{}] → 0, {j, u - k - 1}]]) * Sum[
    Multinomial[i, j, k - i - j] * restrictedMultidimensionalSum[
Multinomial @@ Table[Subscript[k, j], {j, m - u}] * Product[Gamma[Subscript[b, j] -
      Subscript[b, m] - Subscript[ì, m]] * (KroneckerDelta[
      Subscript[k, j]] + UnitStep[Subscript[k, j] - 1] *
BellY[Table1[{1, PolyGamma[t - 1, Subscript[b, j] - Subscript[b, m] - Subscript[ì, m]]},
      {t, Subscript[k, j]}]]), {j, 1, m - u}],
    k, {m - u, i}] * restrictedMultidimensionalSum[
Multinomial @@ Table[Subscript[k, j], {j, n}] * Product[
    Gamma[1 - Subscript[a, j] + Subscript[b, m] + Subscript[ì, m]] *
    (-1)^Subscript[k, j] * (KroneckerDelta[Subscript[k, j]] +
UnitStep[Subscript[k, j] - 1] * BellY[Table1[{1, PolyGamma[t - 1, 1 - Subscript[a, j] +
      Subscript[b, m] + Subscript[ì, m]]},
      {t, Subscript[k, j]}]]), {j, 1, n}], k, {n, j}] *
Derivative[k - i - j][f3][0], {i, 0, k}, {j, 0, k}], {k, 0, u - 1}] /.
    Table1 → Table /. BellY[{}] → 1;
res = res0 /. Derivative[s_.][f3][0] :→ qqq[s, u] /. f3[0] :→ Pi^(-u); res] /.
    Solve[Table[Subscript[b, m - j + 1] + s == -Subscript[ì, m - j + 1] + ϵ, {j, u}],
 Union[{s}, Table[Subscript[b, m - j + 1], {j, 2, u}]]][[1]]}], {u, 1, 6}]]]
TableForm[Simplify[Table[{Ans[[u, 1]], Ans[[u, 2]] / Ans[[u, 3]]}, {u, 1, 6}] /. Residue1 → Residue /.
  Gamma[1 + Subscript[ì, ss_]] :→ Subscript[ì, ss]!]]
```

```
TableForm[
 {{{s → ϵ − Subscript[b, 7] − Subscript[ì, 7]}, 1}, {{s → ϵ − Subscript[b, 7] − Subscript[ì, 7],
      Subscript[b, 6] → Subscript[b, 7] − Subscript[ì, 6] + Subscript[ì, 7]}, 1},
 {{s → ϵ − Subscript[b, 7] − Subscript[ì, 7],
      Subscript[b, 5] → Subscript[b, 7] − Subscript[ì, 5] + Subscript[ì, 7],
      Subscript[b, 6] → Subscript[b, 7] − Subscript[ì, 6] + Subscript[ì, 7]}, 1},
 {{s → ϵ − Subscript[b, 7] − Subscript[ì, 7],
      Subscript[b, 4] → Subscript[b, 7] − Subscript[ì, 4] + Subscript[ì, 7],
      Subscript[b, 5] → Subscript[b, 7] − Subscript[ì, 5] + Subscript[ì, 7],
  Subscript[b, 6] → Subscript[b, 7] − Subscript[ì, 6] + Subscript[ì, 7]}, 1},
 {{s → ϵ − Subscript[b, 7] − Subscript[ì, 7],
      Subscript[b, 3] → Subscript[b, 7] − Subscript[ì, 3] + Subscript[ì, 7],
      Subscript[b, 4] → Subscript[b, 7] − Subscript[ì, 4] + Subscript[ì, 7],
  Subscript[b, 5] → Subscript[b, 7] − Subscript[ì, 5] + Subscript[ì, 7],
      Subscript[b, 6] → Subscript[b, 7] − Subscript[ì, 6] + Subscript[ì, 7]}, 1},
 {{s → ϵ − Subscript[b, 7] − Subscript[ì, 7],
      Subscript[b, 2] → Subscript[b, 7] − Subscript[ì, 2] + Subscript[ì, 7],
      Subscript[b, 3] → Subscript[b, 7] − Subscript[ì, 3] + Subscript[ì, 7],
  Subscript[b, 4] → Subscript[b, 7] − Subscript[ì, 4] + Subscript[ì, 7],
      Subscript[b, 5] → Subscript[b, 7] − Subscript[ì, 5] + Subscript[ì, 7],
  Subscript[b, 6] → Subscript[b, 7] − Subscript[ì, 6] + Subscript[ì, 7]}, 1}}]
```

Case of right u-th order poles

$$\text{Residue}\left[\frac{\left(\prod_{j=1}^{m}\text{Gamma}[b_j+s]\right)\prod_{i=1}^{n}\text{Gamma}[1-a_i-s]}{\left(\prod_{i=n+1}^{p}\text{Gamma}[a_i+s]\right)\prod_{j=m+1}^{q}\text{Gamma}[1-b_j-s]}\,z^{-s},\ \{s,\ 1-a_{n-j+1}+\grave{\imath}_{n-j+1}\}\right] ==$$

$$\frac{z^{\frac{-1+a_n-\grave{\imath}_n}{\alpha_n}}\,(-1)^{u+\sum_{j=1}^{u}\grave{\imath}_{n-j+1}}}{(u-1)\,!}\left(\sum_{k=0}^{u-1}\text{Binomial}[u-1,k]\right.$$

$$\left(\frac{\text{KroneckerDelta}[u-1-k]}{\left(\prod_{i=n-u+1}^{p}\text{Gamma}\left[a_i-\frac{-1+a_n-\grave{\imath}_n}{\alpha_n}\,\alpha_i\right]\right)\prod_{j=m+1}^{q}\text{Gamma}\left[1-b_j+\frac{-1+a_n-\grave{\imath}_n}{\alpha_n}\,\beta_j\right]}+\text{UnitStep}[u-k-2]\right.$$

$$\text{BellY}\left[\text{Table}\left[\left\{(-1)^j\,j\,!\left(\left(\prod_{i=n-u+1}^{p}\text{Gamma}\left[a_i-\frac{-1+a_n-\grave{\imath}_n}{\alpha_n}\,\alpha_i\right]\right)\prod_{j=m+1}^{q}\text{Gamma}\left[1-b_j+\frac{-1+a_n-\grave{\imath}_n}{\alpha_n}\right.\right.\right.\right.\right.$$

$$\left.\left.\beta_j\right]\right)^{-1-j},\ \sum_{i=0}^{j}\text{Binomial}[j,i]\left(\sum_{k_{m+1}=0}^{j-i}\ldots\sum_{k_q=0}^{j-i}\text{KroneckerDelta}\left[j-i,\ \sum_{j=m+1}^{q}k_j\right]\right.$$

$$\text{Multinomial}[k_{m+1},\ldots,k_q]\prod_{j=m+1}^{q}\left(\text{Gamma}\left[1-b_j+\frac{-1+a_n-\grave{\imath}_n}{\alpha_n}\,\beta_j\right]\right.$$

$$(-\beta_j)^{k_j}\Big(\text{KroneckerDelta}[k_j] + \text{UnitStep}[k_j - 1]\ \text{BellY}\Big[\text{Table}\Big[\Big\{1, \text{PolyGamma}\Big[t-1, 1-b_j + \frac{-1+a_n-\grave{\imath}_n}{\alpha_n}\beta_j\Big]\Big\}, \{t, k_j\}\Big]\Big]\Big)\Big)\Big)$$

$$\Bigg(\sum_{k_{n-u+1}=0}^{i} \cdots \sum_{k_p=0}^{i} \text{KroneckerDelta}\Big[i, \sum_{j=n-u+1}^{p} k_j\Big]\ \text{Multinomial}[k_{n-u+1}, \ldots, k_p]$$

$$\prod_{j=n-u+1}^{p}\Big(\text{Gamma}\Big[a_j - \frac{-1+a_n-\grave{\imath}_n}{\alpha_n}\alpha_j\Big]\alpha_j^{k_j}$$

$$\Big(\text{KroneckerDelta}[k_j] + \text{UnitStep}[k_j-1]\ \text{BellY}\Big[\text{Table}\Big[\Big\{1, \text{PolyGamma}\Big[t-1, a_j - \frac{-1+a_n-\grave{\imath}_n}{\alpha_n}\alpha_j\Big]\Big\}, \{t, k_j\}\Big]\Big]\Big)\Big)\Big\}, \{j, u-k-1\}\Big]\Big]\Bigg)$$

$$\sum_{i=0}^{k}\sum_{j=0}^{k}\Bigg(\text{Multinomial}[i, j, k-i-j]\ \text{Piecewise}\Big[\{\{1, m === u\}\}, \Big(\sum_{k_1=0}^{i}\cdots\sum_{k_m=0}^{i}\text{KroneckerDelta}\Big[$$

$$i, \sum_{j=1}^{m}k_j\Big]\ \text{Multinomial}[k_1, \ldots, k_m]$$

$$\prod_{j=1}^{m}\Big(\text{Gamma}\Big[b_j - \frac{-1+a_n-\grave{\imath}_n}{\alpha_n}\beta_j\Big]\beta_j^{k_j}\Big(\text{KroneckerDelta}[k_j] + \text{UnitStep}[k_j-1]\ \text{BellY}\Big[$$

$$\text{Table}\Big[\Big\{1, \text{PolyGamma}\Big[t-1, b_j - \frac{-1+a_n-\grave{\imath}_n}{\alpha_n}\beta_j\Big]\Big\}, \{t, k_j\}\Big]\Big]\Big)\Big)\Big)\Big]$$

$$\Bigg(\sum_{k_1=0}^{j}\cdots\sum_{k_{n-u}=0}^{j}\text{KroneckerDelta}\Big[j, \sum_{j=1}^{n-u}k_j\Big]\ \text{Multinomial}[k_1, \ldots, k_{n-u}]\prod_{j=1}^{n-u}\text{Gamma}\Big[$$

$$1 - a_j + \frac{-1+a_n-\grave{\imath}_n}{\alpha_n}\alpha_j\Big](-\alpha_j)^{k_j}\Big(\text{KroneckerDelta}[k_j] + \text{UnitStep}[k_j-1]$$

$$\text{BellY}\Big[\text{Table}\Big[\Big\{1, \text{PolyGamma}\Big[t-1, 1-a_j + \frac{-1+a_n-\grave{\imath}_n}{\alpha_n}\alpha_j\Big]\Big\}, \{t, k_j\}\Big]\Big]\Big)\Big)$$

$$(k-i-j)!\sum_{r=0}^{\text{Floor}\left[\frac{k-i-j}{2}\right]}\frac{(-\text{Log}[z])^{k-i-j-2r}\pi^{2r}}{(k-i-j-2r)!}\Big(\delta_r + \frac{\text{UnitStep}[r-1]}{r!}\ \text{BellY}\Big[$$

$$\text{Table}\Big[\Big\{(-1)^{i}\ \text{Pochhammer}[-u, i],$$

$$\frac{(-1)^{i-1}\,2\,(2^{2i-1}-1)\,i!\ \text{BernoulliB}[2i]}{(2i)!}\Big\}, \{i, r\}\Big]\Big]\Big)\Big)\Bigg) /;$$

u ∈ Integers&& $\imath_{n-j+1}$ ∈ Integers&& $\imath_{n-j+1} \geq 0$ && $1 \leq j \leq u \leq n$ &&

$a_{n-j+1} == \frac{(-1 + a_n - \imath_n)\ \alpha_{n-j+1}}{\alpha_n} + 1 + \imath_{n-j+1}$ &&

$0 \leq j \leq u \leq n$ &&

Not$\left[a_i + \frac{1 - a_n + \imath_n}{\alpha_n}\ \alpha_i \in \text{Integers} \&\&\ a_i + \frac{1 - a_n + \imath_n}{\alpha_n}\ \alpha_i \leq 0 \ \&\&\ n + 1 \leq i \leq p\right]$ &&

Not$\left[1 - b_j - \frac{1 - a_n + \imath_n}{\alpha_n}\ \beta_j \in \text{Integers} \&\&\ 1 - b_j - \frac{1 - a_n + \imath_n}{\alpha_n}\ \beta_j \leq 0 \ \&\&\ m + 1 \leq j \leq q\right]$

$$\operatorname{res}\left(\frac{\left(\prod_{j=1}^{m} \Gamma(b_j + s)\right) \prod_{i=1}^{n} \Gamma(1 - a_i - s)}{\left(\prod_{i=n+1}^{p} \Gamma(a_i + s)\right) \prod_{j=m+1}^{q} \Gamma(1 - b_j - s)}\ z^{-s},\ \{s, -b_m - \imath_m\}\right) =$$

$$\frac{z^{b_m + \imath_m}\ (-1)^{\sum_{j=1}^{u} \imath_{m-j+1}}}{(u-1)!} \sum_{k=0}^{u-1} \binom{u-1}{k} \left(\frac{\delta_{u-k-1}}{\left(\prod_{i=n+1}^{p} \Gamma(a_i - b_m - \imath_m)\right) \prod_{j=m-u+1}^{q} \Gamma(1 - b_j + b_m + \imath_m)} + \right.$$

$$\theta(u - k - 2)\ \text{BellY}\Big[\text{Table}\Big[\Big\{(-1)^j\ j!\ \left(\left(\prod_{i=n+1}^{p} \Gamma(a_i - b_m - \imath_m)\right) \prod_{j=m-u+1}^{q} \Gamma(1 - b_j + b_m + \imath_m)\right)^{-j-1},$$

$$\sum_{i=0}^{j} \binom{j}{i} \left(\sum_{k_{m-u+1}=0}^{j-i} \ldots \sum_{k_q=0}^{j-i} \delta_{j-i, \sum_{j=m-u+1}^{q} k_j}\ (k_{m-u+1} + \ldots + k_q;\ k_{m-u+1},\ \ldots,\ k_q) \prod_{j=m-u+1}^{q} \Gamma(1 - b_j + b_m + \imath_m)\right.$$

$$\left.(-\beta_j)^{k_j}\ \left(\delta_{k_j} + \theta(k_j - 1)\ \text{BellY}\left[\text{Table}\left[\left\{1,\ \psi^{(t-1)}(1 - b_j + b_m + \imath_m)\right\},\ \{t,\ k_j\}\right]\right]\right)\right)$$

$$\sum_{k_{n+1}=0}^{i} \ldots \sum_{k_p=0}^{i} \delta_{i, \sum_{j=n+1}^{p} k_j}\ (k_{n+1} + \ldots + k_p;\ k_{n+1},\ \ldots,\ k_p) \prod_{j=n+1}^{p} \Gamma(a_i - b_m - \imath_m)$$

$$\left(\delta_{k_j}+\theta(k_j-1)\,\text{BellY}\left[\text{Table}\left[\left\{1,\psi^{(t-1)}(a_i-b_m-\grave{\imath}_m)\right\},\{t,k_j\}\right]\right]\right\},\{j,u-k-1\}\Big]\Big]\Bigg)$$

$$\sum_{i=0}^{k}\sum_{j=0}^{k}(k;i,j,k-i-j)\left\{\begin{array}{l}1\\ \sum_{k_1=0}^{i}\ldots\sum_{k_{m-u}=0}^{i}\delta_{i,\sum_{j=1}^{m-u}k_j}(k_1+\ldots+k_{m-u};k_1,\ldots,k_{m-u})\\ \prod_{j=1}^{m-u}\Gamma(b_j-b_m-\grave{\imath}_m)\left(\delta_{k_j}+\theta(k_j-1)\,\text{BellY}\left[\text{Table}\left[\left\{1,\psi^{(t-1)}(b_j-b_m-\grave{\imath}_m)\right\},\{t,k_j\}\right]\right]\right)\end{array}\right.$$

$$(-\alpha_j)^{k_j}\left(\delta_{k_j}+\theta(k_j-1)\,\text{BellY}\left[\text{Table}\left[\left\{1,\psi^{(t-1)}(1-a_j+b_m+\grave{\imath}_m)\right\},\{t,k_j\}\right]\right]\right)\Bigg)(k-i-j)!$$

$$\sum_{r=0}^{\left\lfloor\frac{k-i-j}{2}\right\rfloor}\frac{(-\log(z))^{k-i-j-2r}\,\pi^{2r}}{(k-i-j-2r)!}\left(\delta_r+\frac{\theta(r-1)}{r!}\,\text{BellY}\left[\text{Table}\left[\left\{(-1)^i\,(-u)_i,\frac{(-1)^{i-1}\,2\,(2^{2i-1}-1)\,i!\,B_{2i}}{(2i)!}\right\},\{i,r\}\right]\right]\right)\Bigg)\Bigg)/;$$

$$u\in\mathbb{N}\land\grave{\imath}_{m-j+1}\in\mathbb{N}\land 1\le j\le u\le m\land b_{m-j+1}=b_m+\grave{\imath}_m-\grave{\imath}_{m-j+1}\land 0\le j\le u\le m\land$$
$$\neg\,(a_i-b_m-\grave{\imath}_m\in\mathbb{Z}\land a_i-b_m-\grave{\imath}_m\le 0\land n+1\le i\le p)\land$$
$$\neg\left(b_m+\grave{\imath}_m-b_j+1\in\mathbb{Z}\land b_m+\grave{\imath}_m-b_j+1\le 0\land m+1\le j\le q\right)$$

```
Clear[restrictedMultidimensionalSum]; restrictedMultidimensionalSum[body_, k_, {Q_, M_}] :=
  Module[{body1, kIterators},
    body1 = body /. Subscript[k, Q] → M - Sum[Subscript[k, j], {j, 1, Q - 1}];
   kIterators = Table[{Subscript[k, j], 0, M - Sum[Subscript[k, i], {i, 1, j - 1}]}, {j, Q - 1}];
   If[Q === 1, body /. Subscript[k, j_] :→ M,
      Sum[Evaluate[body1], Evaluate[Sequence @@ kIterators]]]]

Ans = Quiet[Simplify[Table[With[{m = 1, n = 7, p = 8, q = 2},
          {Solve[Table[1 - Subscript[a, n - j + 1] - s == -Subscript[ì, n - j + 1] - ϵ, {j, u}],
     Union[{s}, Table[Subscript[a, n - j + 1], {j, 2, u}]]][[1]],
             Simplify[Residue1[(Product[Gamma[Subscript[b, j] + s], {j, 1, m}] *
                          Product[Gamma[1 - Subscript[a, i] - s], {i, 1, n}]) /
      (Product[Gamma[Subscript[a, i] + s], {i, n + 1, p}] *
                          Product[Gamma[1 - Subscript[b, j] - s], {j, m + 1, q}]) / z^s, {ϵ, 0},
     Assumptions → {And @@ Flatten[Union[Table[{Element[Subscript[ì, n - j + 1], Integers]}, {j, 1,
                                   u}], Table[{Subscript[ì, n - j + 1] ≥ 0}, {j, 1, u}]]]}]] /.
    Solve[Table[1 - Subscript[a, n - j + 1] - s == -Subscript[ì, n - j + 1] - ϵ, {j, u}],
                  Union[{s}, Table[Subscript[a, n - j + 1], {j, 2, u}]]][[1]],
    z^(-1 + Subscript[a, n] - Subscript[ì, n]) *
                 (-1)^(u + Sum[Subscript[ì, n - j + 1], {j, 1, u}]) * Pi^u * Module[{qqq, res0, res},
```

```
qqq[k_, u_] := (k! * Sum[((-Log[z])^(k - 2*r) / (k - 2*r)!) * Pi^(2*r) * (KroneckerDelta[
        r] + (UnitStep[r - 1] / r!) * BellY[Table[{(-1)^i * Pochhammer[
             -u, i], ((-1)^(i - 1) * 2 * (2^(2*i - 1) - 1) * i! *
      BernoulliB[2*i]) / (2*i)!}, {i, 1, r}]]), {r, 0, Floor[k/2]}]) / Pi^u;
res0 = (1 / (u - 1)!) * Sum[Binomial[u - 1, k] *
      (KroneckerDelta[u - 1 - k] / (Product[Gamma[1 + Subscript[a, j] -
            Subscript[a, n] + Subscript[ì, n]], {j, n - u + 1, p}] *
    Product[Gamma[-Subscript[b, j] + Subscript[a, n] - Subscript[ì, n]], {j, m + 1, q}]) +
        UnitStep[u - k - 2] * BellY[Table[
    {(-1)^j * j! * (Product[Gamma[1 + Subscript[a, j] - Subscript[a, n] + Subscript[ì, n]], {j,
              n - u + 1, p}] * Product[Gamma[-Subscript[b, j] +
              Subscript[a, n] - Subscript[ì, n]],
      {j, m + 1, q}])^(-1 - j), Sum[Binomial[j, i] * restrictedMultidimensionalSum[
            Multinomial @@ Table[Subscript[k, j], {j, p - n + u}] *
             Product[Gamma[1 + Subscript[a, j] -
       Subscript[a, n] + Subscript[ì, n]] * (KroneckerDelta[Subscript[k, j - n + u]] +
              BellY[Table1[{1, PolyGamma[-1 + t,
              1 + Subscript[a, j] - Subscript[a, n] + Subscript[ì,
        n]]}, {t, Subscript[k, j - n + u]}]]), {j, n - u + 1, p}], k, {p - n + u, i}] *
          restrictedMultidimensionalSum[
            Multinomial @@ Table[Subscript[k, j], {j, q - m}] *
      Product[Gamma[-Subscript[b, j] + Subscript[a, n] - Subscript[ì, n]] * (-1)^Subscript[k,
              j - m] * (KroneckerDelta[Subscript[k, j - m]] +
              BellY[Table1[{1, PolyGamma[t - 1,
        -Subscript[b, j] + Subscript[a, n] - Subscript[ì, n]]}, {t, Subscript[k, j - m]}]]),
              {j, m + 1, q}], k, {q - m, j - i}],
          {i, 0, j}]} /. Table1 → Table /.
    BellY[{}] → 0, {j, u - k - 1}]]) * Sum[
        Multinomial[i, j, k - i - j] * restrictedMultidimensionalSum[
          Multinomial @@ Table[Subscript[k, j], {j, m}] *
    Product[Gamma[1 + Subscript[b, j] - Subscript[a, n] + Subscript[ì, n]] *
              (KroneckerDelta[Subscript[k, j]] + UnitStep[Subscript[k, j] -
                1] * BellY[Table1[{1, PolyGamma[t - 1,
      1 + Subscript[b, j] - Subscript[a, n] + Subscript[ì, n]]}, {t, Subscript[k, j]}]]),
            {j, 1, m}], k, {m, i}] * restrictedMultidimensionalSum[
    Multinomial @@ Table[Subscript[k, j], {j, n - u}] * Product[
            Gamma[-Subscript[a, j] + Subscript[a, n] - Subscript[ì, n]] *
              (-1)^Subscript[k, j] * (KroneckerDelta[Subscript[k, j]] +
    UnitStep[Subscript[k, j] - 1] * BellY[Table1[{1, PolyGamma[t - 1, -Subscript[a, j] +
              Subscript[a, n] - Subscript[ì, n]]}, {t, Subscript[
```

```
                                    k, j]}]]), {j, 1, n - u}], k, {n - u, j}] *
      Derivative[k - i - j][f3][0], {i, 0, k}, {j, 0, k}], {k, 0, u - 1}] /.
                     Table1 → Table /. BellY[{}] → 1;
    res = res0 /. Derivative[s_.][f3][0] :→ qqq[s, u] /. f3[0] :→ Pi^(-u); res] /.
          Solve[Table[1 - Subscript[a, n - j + 1] - s == -Subscript[ì, n - j + 1] - ϵ, {j, u}],
    Union[{s}, Table[Subscript[a, n - j + 1], {j, 2, u}]]][[1]]}], {u, 1, 6}]]]
TableForm[FullSimplify[Table[{Ans[[u, 1]], Ans[[u, 2]]/Ans[[u, 3]]}, {u, 1, 6}] /. Residue1 → Residue]]
TableForm[{{{s → 1 + ϵ - Subscript[a, 7] + Subscript[ì, 7]}, 1},
   {{s → 1 + ϵ - Subscript[a, 7] + Subscript[ì, 7],
      Subscript[a, 6] → Subscript[a, 7] + Subscript[ì, 6] - Subscript[ì, 7]}, 1},
  {{s → 1 + ϵ - Subscript[a, 7] + Subscript[ì, 7],
      Subscript[a, 5] → Subscript[a, 7] + Subscript[ì, 5] - Subscript[ì, 7],
  Subscript[a, 6] → Subscript[a, 7] + Subscript[ì, 6] - Subscript[ì, 7]}, 1},
  {{s → 1 + ϵ - Subscript[a, 7] + Subscript[ì, 7],
      Subscript[a, 4] → Subscript[a, 7] + Subscript[ì, 4] - Subscript[ì, 7],
  Subscript[a, 5] → Subscript[a, 7] + Subscript[ì, 5] - Subscript[ì, 7],
      Subscript[a, 6] → Subscript[a, 7] + Subscript[ì, 6] - Subscript[ì, 7]}, 1},
  {{s → 1 + ϵ - Subscript[a, 7] + Subscript[ì, 7],
      Subscript[a, 3] → Subscript[a, 7] + Subscript[ì, 3] - Subscript[ì, 7],
  Subscript[a, 4] → Subscript[a, 7] + Subscript[ì, 4] - Subscript[ì, 7],
      Subscript[a, 5] → Subscript[a, 7] + Subscript[ì, 5] - Subscript[ì, 7],
  Subscript[a, 6] → Subscript[a, 7] + Subscript[ì, 6] - Subscript[ì, 7]}, 1},
  {{s → 1 + ϵ - Subscript[a, 7] + Subscript[ì, 7],
      Subscript[a, 2] → Subscript[a, 7] + Subscript[ì, 2] - Subscript[ì, 7],
  Subscript[a, 3] → Subscript[a, 7] + Subscript[ì, 3] - Subscript[ì, 7],
      Subscript[a, 4] → Subscript[a, 7] + Subscript[ì, 4] - Subscript[ì, 7],
  Subscript[a, 5] → Subscript[a, 7] + Subscript[ì, 5] - Subscript[ì, 7],
      Subscript[a, 6] → Subscript[a, 7] + Subscript[ì, 6] - Subscript[ì, 7]}, 1}}]
```

Fox-H transform as case of integral transforms

$$\mathcal{H}[f[t], t, x] == \int_0^\infty H_{p,q}^{m,n}\left[x\,t \,\middle|\, \begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix}\right] f[t]\, dt$$

Below we present formulas which defined about 100 integral transforms: Abel transform, Bessel **H**-transform (StruveTransform), Bessel Y-transform (NeumannTransform), Buschman transform, FourierTransform, FourierCosTransform, FourierSinTransform, Fourier-Stieltjes transform, G-transform, generalized Laplace transform, HankelTransform, Hartley transform, Hermite transform, Hilbert transform, Kontorovich-Lebedev transform, LaplaceTransform, Mehler–Fock transform, Generalized Mehler–Fock transform, Meijer transform, MellinTransform, Narain G-transform, Olevski transform, RadonTransform, Riemann-Liouville-transform (FractionalD), Riesz transform, Stieltjes transform, Two-sided Laplace transform, Weierstrass transform, W-transform and Weyl transform. Also for many of them we present their generalizations with parameter α included in factor t^{α} and give representation formulas through the more general GTransform with function MeijerG in the kernel.

Integral transforms, big picture

In[∘]:=

```
Information["*Transform*"]
```

Out[∘]=

System`

AffineTransform	FindRegionTransform	InverseBilateralLaplaceTransform	LinearFractionalTransform	StationaryWaveletPacketTransform
AudioSpectralTransformation	FourierCosTransform	InverseBilateralZTransform	LinearizingTransformationData	StationaryWaveletTransform
BilateralLaplaceTransform	FourierSequenceTransform	InverseContinuousWaveletTransform	ListFourierSequenceTransform	TopHatTransform
BilateralZTransform	FourierSinTransform	InverseDistanceTransform	ListZTransform	TransferFunctionTransform
BottomHatTransform	FourierTransform	InverseFourierCosTransform	MellinTransform	TransformationClass
ConfidenceTransform	GeometricTransformation	InverseFourierSequenceTransform	MorphologicalTransform	TransformationFunction
ContinuousWaveletTransform	GeometricTransformation3DBox	InverseFourierSinTransform	NondimensionalizationTransform	TransformationFunctions
CoordinateTransform	GeometricTransformation3DBoxOptions	InverseFourierTransform	RadonTransform	TransformationMatrix
CoordinateTransformData	GeometricTransformationBox	InverseHankelTransform	ReflectionTransform	TransformedDistribution
DirichletTransform	GeometricTransformationBoxOptions	InverseLaplaceTransform	RescalingTransform	TransformedField
DiscreteChirpZT…		InverseMellinTra…	RotationTransfo…	TransformedPro…

ransform	HankelTransform	nsform	rm	cess
DiscreteHadamardTransform	HistogramTransform	InverseRadonTransform	ScalingTransform	TransformedRegion
DiscreteWaveletPacketTransform	HistogramTransformInterpolation	InverseTransformedRegion	ShearingTransform	TranslationTransform
DiscreteWaveletTransform	HitMissTransform	InverseWaveletTransform	SkeletonTransform	ZTransform
DistanceTransform	ImageForwardTransformation	InverseZTransform	SpatialTransformationLayer	
FillingTransform	ImagePerspectiveTransformation	LaplaceTransform	StateSpaceTransform	
FindGeometricTransform	ImageTransformation	LiftingWaveletTransform	StateTransformationLinearize	

$$\mathcal{K}[f[t],t,x] == \int_a^b K[x,t] \times f[t] \, dt == g[x]$$

$$\mathcal{K}^{-1}[g[x],x,t] == f[t] \quad (*??*)$$

convolution transforms or index transforms or other

$$\mathcal{K}_1[f[t],t,x] == \int_0^\infty K_1[x\,t]\, f[t]\, dt == g[x]$$

$$\mathcal{K}_2[f[t],t,x] == \int_0^\infty K_2\left[\frac{x}{t}\right] f[t]\, dt == g[x] \qquad (*\mathcal{K}_2[f[t],t,x]*)$$

$$\mathcal{K}_3[f[t],t,x] == \int_0^\infty K[x-t] \times f[t]\, dt == g[x]$$

Information[Convolve]

Out[•]=

Symbol

Convolve[f, g, x, y] gives the convolution with respect to x of the expressions f and g.

Convolve[f, g, {x_1, x_2, ...}, {y_1, y_2, ...}] gives the multidimensional convolution.

Documentation Web »

Attributes {Protected, ReadProtected}

Full Name System`Convolve

Some examples of integral transforms

$$\text{FourierTransform}[f[t], t, x] == \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f[t]\, e^{i t x}\, dt$$

$$\text{LaplaceTransform}[f[t], t, x] == \int_0^{\infty} e^{-x t} f[t]\, dt == g[x] \quad (*K[y]==e^{-y}*)$$

$$\text{LaplaceTransform}[f[t], t, x] == \sqrt{2\pi}\ \text{FourierTransform}[\text{UnitStep}[t]\, f[t], t, i\, x]$$

$$\text{HankelTransform}[f[t], t, x, v] ==$$

$$\int_0^{\infty} \text{BesselJ}[v, x\, t]\, f[t]\, t\, dt == g[x] \quad (*K[y]==\text{BesselJ}[v,y]*)$$

$$\text{FourierCosTransform}[f[t], t, x] ==$$

$$\sqrt{\frac{2}{\pi}} \int_0^{\infty} \text{Cos}[x\, t]\, f[t]\, dt == g[x] \quad \left(*K[y]==\sqrt{\tfrac{2}{\pi}}\,\text{Cos}[y]*\right)$$

$$\text{MellinTransform}[f[t], t, x] == \int_0^{\infty} t^{x-1} f[t]\, dt == g[x] \quad (*K[x,t]==t^{x-1}*)$$

$$\text{MellinTransform}[f[t], t, x] == \sqrt{2\pi}\ \text{FourierTransform}\left[f\left[e^{-t}\right], t, i\, x\right]$$

$$\text{InverseMellinTransform}[F[x], x, t] == \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} F[x]\, t^{-x}\, dx \quad (*==f[t]*)$$

$$\text{MellinTransform}\left[\int_0^{\infty} f[t] * g\left[\frac{x}{t}\right] \frac{1}{t}\, dt, t, x\right] ==$$

$$\text{MellinTransform}[f[t], t, x] * \text{MellinTransform}[g[t], t, x]$$

$$\text{MellinTransform}\left[t^a \int_0^{\infty} \tau^{b-1} f[\tau^c] \times g[t\,\tau^e]\, d\tau, t, x\right] ==$$

$$\frac{1}{\text{Abs}[c]}\ \text{MellinTransform}\left[f[t], t, \frac{b - e\,(a + x)}{c}\right] \times \text{MellinTransform}[g[t], t, a + x]\ \ /;$$

$$c \neq 0 \land c \in \text{Reals} \land e \in \text{Reals}$$

$$\int_0^{\infty} \tau^{b-1} f[\tau^c] \times g[t\,\tau^e]\, d\tau ==$$

$$\frac{t^{-a}}{\text{Abs}[c]\ 2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \left(\text{MellinTransform}\left[f[t], t, \frac{b - e\,(a + x)}{c}\right] \times \text{MellinTransform}[g[t], t, a + x]\right) t^{-x}\, dx$$

Inverse Mellin transforms

06.05.22.0001.01

$$\text{InverseMellinTransform}[\text{Gamma}[s], s, t] == e^{-t} \; /; \text{Re}[s] > 0$$

06.05.22.0002.01

$$\text{InverseMellinTransform}[\text{Gamma}[s]\,\text{Gamma}[a-s], s, t] == (1+t)^{-a}\,\text{Gamma}[a] \; /; \; 0 < \text{Re}[s] < \text{Re}[a]$$

06.05.22.0003.01

$$\text{InverseMellinTransform}\left[\frac{\text{Gamma}[s]}{\text{Gamma}[a-s]}, s, t\right] == t^{\frac{1-a}{2}}\,\text{BesselJ}\left[a-1, 2\sqrt{t}\right] \; /; \; 0 < \text{Re}[s] < \frac{2\,\text{Re}[a]+1}{4}$$

06.05.22.0004.01

$$\text{InverseMellinTransform}\left[\frac{\text{Gamma}[s]}{\text{Gamma}[a+s]}, s, t\right] == \frac{(1-t)^{a-1}\,\text{UnitStep}[1-t]}{\text{Gamma}[a]} \; /; \; \text{Re}[a] > 0 \land \text{Re}[s] > 0$$

06.05.22.0005.01

$$\text{InverseMellinTransform}\left[\frac{\prod_{k=1}^{A}\text{Gamma}[a_k+s]\,\prod_{k=1}^{B}\text{Gamma}[b_k-s]}{\prod_{k=1}^{C}\text{Gamma}[c_k+s]\,\prod_{k=1}^{D}\text{Gamma}[d_k-s]}, s, t\right] ==$$
$$\text{MeijerG}[\{\{1-b_1, \ldots, 1-b_B\}, \{c_1, \ldots, c_C\}\}, \{\{a_1, \ldots, a_A\}, \{1-d_1, \ldots, 1-d_D\}\}, t] \; /;$$
$$\Delta == A + D - B - C \land E == A + B - C - D \land \nu == \sum_{k=1}^{A} a_k + \sum_{k=1}^{B} b_k - \sum_{k=1}^{C} c_k - \sum_{k=1}^{D} d_k \land$$
$$-\text{Min}[\text{Re}[a_1], \ldots, \text{Re}[a_A]] < \text{Re}[s] < \text{Min}[\text{Re}[b_1], \ldots, \text{Re}[b_B]] \land$$
$$\Bigg(\left(\text{Abs}[\text{Arg}[t]] < \frac{\pi E}{2} \land E > 0\right) \lor$$
$$\left(\text{Abs}[\text{Arg}[t]] == \frac{\pi E}{2} \land E > 0 \land \Delta\,\text{Re}[s] + \text{Re}[\nu] - \frac{E}{2} < -1\right) \lor$$
$$\left(t > 0 \land E == 0 \land \Delta \neq 0 \land \Delta\,\text{Re}[s] + \text{Re}[\nu] < \frac{1}{2}\right) \lor$$
$$(t > 0 \land E == 0 \land \Delta == 0 \land ((\text{Re}[\nu] < 0 \land t \neq 1) \lor (\text{Re}[\nu] < -1 \land t == 1)))\Bigg)$$

More examples of Fox H-transform (in Mathematica)

```
GenericIntegralTransform[f[x], x, z,
  {"H", {{{a1, α1}, …, {an, αn}}, {{an+1, αn+1}, …, {ap, αp}}},
   {{{b1, β1}, …, {bm, βm}}, {{bm+1, βm+1}, …, {bq, βq}}}}]
```

In[•]:=

```
GenericIntegralTransform[Exp[-a x^Catalan], x, z, {"H", {{}, {}}, {{{0, 1}}, {}}}]
```

Out[•]=

$$\frac{\text{FoxH}[\{\{\{0, \text{Catalan}\}\}, \{\}\}, \{\{\{0, 1\}\}, \{\}\}, a\, z^{-\text{Catalan}}]}{z}$$

In[•]:=

```
GenericIntegralTransform[Exp[-a x^Catalan],
  x, z, {"H", {{}, {{β, 1}}}, {{{0, 1}}, {}}}]
```

Out[•]=

$$\frac{\text{FoxH}[\{\{\{0, \text{Catalan}\}\}, \{\}\}, \{\{\{0, 1\}\}, \{\{-\beta, \text{Catalan}\}\}\}, a\, z^{-\text{Catalan}}]}{z}$$

In[•]:=

```
GenericIntegralTransform[Exp[-a x^Catalan], x, z, {"H", {{{β, 1}}, {}}, {{}, {{0, 1}}}}]
```

Out[•]=

$$\frac{\text{FoxH}[\{\{\}, \{\{0, \text{Catalan}\}\}\}, \{\{\{0, 1\}, \{-\beta, \text{Catalan}\}\}, \{\}\}, a\, z^{-\text{Catalan}}]}{z}$$

In[•]:=

```
With[{β = Random[], a = Random[], z = Random[], r = 3/5}, {NIntegrate[
    Exp[-a x^r] (1 - x z)^(β-1) UnitStep[1 - Abs[x z]], {x, 0, 1/z}], Gamma[β]
    FoxH[{{{0, r}}, {}}, {{{0, 1}}, {{-β, r}}}, a z^-r] / z}] // Activate // Chop
```

Out[•]=

{1.11603, 1.11603}

In[∘]:=

```
With[{β = Random[], a = Random[], z = Random[], r = Catalan}, {NIntegrate[
    Exp[-a x^r] (1 - x z)^(β-1) UnitStep[1 - Abs[x z]], {x, 0, 1/z}], Gamma[β]
    FoxH[{{{0, r}}, {}}, {{{0, 1}}, {{-β, r}}}, a z^-r]/z}] // Activate // Chop
```

Out[∘]=

{9.83538, 9.83538}

In[∘]:=

```
With[{β = Random[], a = Random[], z = Random[], r = 1/5}, {NIntegrate[
    Exp[-a x^r] (x z - 1)^(β-1) UnitStep[Abs[x z] - 1], {x, 1/z, ∞}], Gamma[β]
    FoxH[{{}, {{0, r}}}, {{{0, 1}, {-β, r}}, {}}, a z^-r]/z}] // Activate // Chop
```

Out[∘]=

{162646., 162646.}

In[∘]:=

```
With[{β = Random[], a = Random[], z = Random[], r = Catalan/3}, {NIntegrate[
    Exp[-a x^r] (x z - 1)^(β-1) UnitStep[Abs[x z] - 1], {x, 1/z, ∞}], Gamma[β]
    FoxH[{{}, {{0, r}}}, {{{0, 1}, {-β, r}}, {}}, a z^-r]/z}] // Activate // Chop
```

Out[∘]=

{9819.7, 9819.15}

In[∘]:=

```
Table[{Integrate[Exp[-a x^n] (1 - x z)^(β-1), {x, 0, 1/z}, GenerateConditions→ False],
   Gamma[β] z^(1-β) ResourceFunction["FractionalOrderD"][Exp[-a z^n],
    {z, -β}] /. z → 1/z}, {n, 2, 5}] // Activate // PowerExpand
```

Out[∘]=

$$\left\{\left\{\frac{\text{HypergeometricPFQ}\left[\left\{\frac{1}{2},1\right\},\left\{\frac{1}{2}+\frac{\beta}{2},1+\frac{\beta}{2}\right\},-\frac{a}{z^2}\right]}{z\,\beta},\right.\right.$$
$$\left.\frac{2^{-\beta}\sqrt{\pi}\,\text{Gamma}[\beta]\,\text{HypergeometricPFQRegularized}\left[\left\{\frac{1}{2},1\right\},\left\{1+\frac{1}{2}(-1+\beta),1+\frac{\beta}{2}\right\},-\frac{a}{z^2}\right]}{z}\right\},$$
$$\left\{\frac{\text{HypergeometricPFQ}\left[\left\{\frac{1}{3},\frac{2}{3},1\right\},\left\{\frac{1}{3}+\frac{\beta}{3},\frac{2}{3}+\frac{\beta}{3},1+\frac{\beta}{3}\right\},-\frac{a}{z^3}\right]}{z\,\beta},\frac{1}{z}\right.$$
$$3^{-\beta}\,\text{Gamma}\left[\frac{1}{3}\right]\text{Gamma}\left[\frac{2}{3}\right]\text{Gamma}[\beta]$$
$$\left.\text{HypergeometricPFQRegularized}\left[\left\{\frac{1}{3},\frac{2}{3},1\right\},\left\{1+\frac{1}{3}(-2+\beta),1+\frac{1}{3}(-1+\beta),1+\frac{\beta}{3}\right\},-\frac{a}{z^3}\right]\right\},$$
$$\left\{\frac{\text{HypergeometricPFQ}\left[\left\{\frac{1}{4},\frac{1}{2},\frac{3}{4},1\right\},\left\{\frac{1}{4}+\frac{\beta}{4},\frac{1}{2}+\frac{\beta}{4},\frac{3}{4}+\frac{\beta}{4},1+\frac{\beta}{4}\right\},-\frac{a}{z^4}\right]}{z\,\beta},\frac{1}{z}\right.$$
$$4^{-\beta}\sqrt{\pi}\,\text{Gamma}\left[\frac{1}{4}\right]\text{Gamma}\left[\frac{3}{4}\right]\text{Gamma}[\beta]\,\text{HypergeometricPFQRegularized}\Big[$$
$$\left.\left\{\frac{1}{4},\frac{1}{2},\frac{3}{4},1\right\},\left\{1+\frac{1}{4}(-3+\beta),1+\frac{1}{4}(-2+\beta),1+\frac{1}{4}(-1+\beta),1+\frac{\beta}{4}\right\},-\frac{a}{z^4}\Big]\right\},$$
$$\left\{\frac{\text{HypergeometricPFQ}\left[\left\{\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5},1\right\},\left\{\frac{1}{5}+\frac{\beta}{5},\frac{2}{5}+\frac{\beta}{5},\frac{3}{5}+\frac{\beta}{5},\frac{4}{5}+\frac{\beta}{5},1+\frac{\beta}{5}\right\},-\frac{a}{z^5}\right]}{z\,\beta},\right.$$
$$\frac{1}{z}\,5^{-\beta}\,\text{Gamma}\left[\frac{1}{5}\right]\text{Gamma}\left[\frac{2}{5}\right]\text{Gamma}\left[\frac{3}{5}\right]\text{Gamma}\left[\frac{4}{5}\right]$$
$$\text{Gamma}[\beta]\,\text{HypergeometricPFQRegularized}\Big[\left\{\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5},1\right\},$$
$$\left.\left.\left\{1+\frac{1}{5}(-4+\beta),1+\frac{1}{5}(-3+\beta),1+\frac{1}{5}(-2+\beta),1+\frac{1}{5}(-1+\beta),1+\frac{\beta}{5}\right\},-\frac{a}{z^5}\Big]\right\}\right\}$$

In[∘]:=

```
Part[#, 1]/Part[#, 2] & /@% // FunctionExpand // FullSimplify
```

Out[∘]=

{1, 1, 1, 1}

Special case: {m,n,p,q}=={1,0,0,1} (* e^{-z} *)

In[∘]:=

```
With[{m = 1, n = 0, p = 0, q = 1},
  {FoxH[{Table[{a_i, α_i}, {i, 1, n}], Table[{a_i, α_i}, {i, n + 1, p}]},
     {Table[{b_i, β_i}, {i, 1, m}], Table[{b_i, β_i}, {i, m + 1, q}]},
     z], 1/(2 π ⅈ) ContourIntegrate[
      ((∏_{j=1}^m Gamma[b_j + β_j s]) ∏_{i=1}^n Gamma[1 - a_i - α_i s]) /
      ((∏_{i=n+1}^p Gamma[a_i + α_i s]) ∏_{j=m+1}^q Gamma[1 - b_j - β_j s]) z^-s, {s, ℒ}]}]
```

Out[∘]=

$$\left\{\text{FoxH}[\{\{\},\{\}\},\{\{\{b_1,\beta_1\}\},\{\}\},z],-\frac{i\,\text{ContourIntegrate}[z^{-s}\,\text{Gamma}[b_1+s\,\beta_1],\{s,\mathcal{L}\}]}{2\pi}\right\}$$

Generalized Mellin Parseval relation:

$$\int_0^{\infty}\tau^{b-1} f[\tau^c]\times g[t\,\tau^e]\,d\tau ==$$

$$\frac{t^{-a}}{\text{Abs}[c]\,2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\left(\text{MellinTransform}\left[f[t],t,\frac{b-e\,(a+x)}{c}\right]\times \text{MellinTransform}[g[t],t,a+x]\right)t^{x-1}\,dx$$

$$(*\,H_{0,1}^{1,0}\left(z\,\middle|\,(b_1,\beta_1)\right)==*)\ \text{FoxH}[\{\{\},\{\}\},\{\{\{b_1,\beta_1\}\},\{\}\},z] ==$$

$$\frac{1}{2\pi i}\text{Integrate}[\text{Gamma}[b_1+s\,\beta_1]\,z^{-s},\{\gamma-i\infty,\gamma+i\infty\}]$$

$$H_{0,1}^{1,0}\left(z\,\middle|\,(b_1,\beta_1)\right)$$

In[∘]:=

```
f[z_] := e^-z; g[z_] := e^-z
```

In[∘]:=

```
τ^(b-1) f[τ^c] × g[t τ^e]
```

Out[∘]=

$$e^{-\tau^c-t\,\tau^e}\,\tau^{-1+b}$$

In[∘]:=

```
(MellinTransform[f[t], t, (b - e (a + x))/c] × MellinTransform[g[t], t, a + x])
```

Out[∘]=

$$\text{Gamma}[a+x]\ \text{Gamma}\left[\frac{b-e\ (a+x)}{c}\right]$$

In[∘]:=

```
With[{a = Random[], b = Random[], c = Random[], e = -Random[], t = Random[],
    γ = Random[]}, {NIntegrate[τ^(b-1) f[τ^c] × g[t τ^e], {τ, 0, ∞}],
    NIntegrate[(Gamma[a + x] Gamma[(b-e (a+x))/c]) t^-x, {x, γ - ⅈ ∞, γ + ⅈ ∞}] / (t^a (Abs[c] 2 π ⅈ))}] //
  Activate // Chop
```

Out[∘]=

{1.28253, 1.28253}

In[∘]:=

```
With[{a = Random[], b = Random[], c = Random[], e = Random[], t = Random[],
    γ = Random[]}, {NIntegrate[τ^(b-1) f[τ^c] × g[t τ^e], {τ, 0, ∞}],
    NIntegrate[(Gamma[a + x] Gamma[(b-e (a+x))/c]) t^-x, {x, γ - ⅈ ∞, γ + ⅈ ∞}] / (t^a (Abs[c] 2 π ⅈ))}] //
  Activate // Chop
```

Out[∘]=

{0.604976, 0.604976}

In[∘]:=

```
With[{m = 1, n = 0, p = 0, q = 1},
  {FoxH[{Table[{a_i, α_i}, {i, 1, n}], Table[{a_i, α_i}, {i, n + 1, p}]},
    {Table[{b_i, β_i}, {i, 1, m}], Table[{b_i, β_i}, {i, m + 1, q}]},
   z], 1/(2 π ⅈ) ContourIntegrate[
    ((∏_{j=1}^m Gamma[b_j + β_j s]) ∏_{i=1}^n Gamma[1 - a_i - α_i s]) /
    ((∏_{i=n+1}^p Gamma[a_i + α_i s]) ∏_{j=m+1}^q Gamma[1 - b_j - β_j s]) z^-s, {s, ℒ}]}]
```

```
{FoxH[{{}, {}}, {{{b1, β1}}, {}}, z],
  1/(2 π ⅈ) ContourIntegrate[z^-s Gamma[b1 + β1 s], {s, ℒ}]}

Solve[b1 + β1 s == -k, s]
```

Out[•]=

$$\left\{\left\{s \to \frac{-k - b_1}{\beta_1}\right\}\right\}$$

In[•]:=

```
Residue[z^-s Gamma[b1 + s β1], {s, (-k - b1)/β1},
  Assumptions→ {k ∈ Integers&& k ≥ 0 &&  β1 ∈ Reals&& β1 > 0}]
```

Out[•]=

$$\frac{(-1)^k z^{\frac{k}{\beta_1}+\frac{b_1}{\beta_1}}}{k!\ \beta_1}$$

In[•]:=

```
Sum[%, {k, 0, ∞}]
```

Out[•]=

$$\frac{e^{-z^{\frac{1}{\beta_1}}} z^{\frac{b_1}{\beta_1}}}{\beta_1}$$

```
FoxH[{{}, {}}, {{{b1, β1}}, {}}, z] ==
```

$$\frac{1}{2 \pi i} \text{ContourIntegrate}[z^{-s} \text{Gamma}[b_1 + \beta_1 s], \{s, \mathcal{L}\}] == \frac{e^{-z^{\frac{1}{\beta_1}}} z^{\frac{b_1}{\beta_1}}}{\beta_1}$$

Other {m,n,p,q}

In[∘]:=

```
With[{m = 1, n = 0, p = 0, q = 1},
 {FoxH[{Table[{a_i, α_i}, {i, 1, n}], Table[{a_i, α_i}, {i, n + 1, p}]},
   {Table[{b_i, β_i}, {i, 1, m}], Table[{b_i, β_i}, {i, m + 1, q}]},
   z], 1/(2 π ⅈ) ContourIntegrate[
   ((∏_{j=1}^m Gamma[b_j + β_j s]) ∏_{i=1}^n Gamma[1 - a_i - α_i s]) /
   ((∏_{i=n+1}^p Gamma[a_i + α_i s]) ∏_{j=m+1}^q Gamma[1 - b_j - β_j s]) z^-s, {s, ℒ}]}]
```

Out[∘]=

$$\left\{\text{FoxH}[\{\{\},\{\}\},\{\{\{b_1,\beta_1\}\},\{\}\},z],-\frac{i\ \text{ContourIntegrate}[z^{-s}\,\text{Gamma}[b_1+s\,\beta_1],\{s,\mathcal{L}\}]}{2\pi}\right\}$$

$$H^{m,n}_{p,q}\left[z\,\middle|\,\begin{array}{l}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{array}\right]==H^{m,n}_{p,q}\left(z\,\middle|\,\begin{array}{l}(a_1,\alpha_1),\ldots,(a_n,\alpha_n),\ (a_{n+1},\alpha_{n+1}),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_m,\beta_m),\ (b_{m+1},\beta_{m+1}),\ldots,(b_q,\beta_q)\end{array}\right)\leftrightarrow$$

$$\begin{pmatrix}\overbrace{+\ldots+}^{\text{m times}}(b_j,\beta_j) & \overbrace{-\ldots-}^{\text{n times}}(a_j,\alpha_j)\\ \underbrace{+\ldots+}_{\text{p-n times}}(a_j,\alpha_j) & \underbrace{-\ldots-}_{\text{q-m times}}(b_j,\beta_j)\end{pmatrix}$$

$$\frac{\prod_{j=1}^{m}\text{Gamma}[b_j+\beta_j s]\ \prod_{j=1}^{n}\text{Gamma}[1-a_j-\alpha_j s]}{\prod_{j=n+1}^{p}\text{Gamma}[a_j+\alpha_j s]\ \prod_{j=m+1}^{q}\text{Gamma}[1-b_j-\beta_j s]}\,x^{-s}\leftrightarrow$$

$$\begin{pmatrix}\overbrace{+\ldots+}^{\text{m times}}(b_j,\beta_j) & \overbrace{-\ldots-}^{\text{n times}}(a_j,\alpha_j)\\ \underbrace{+\ldots+}_{\text{p-n times}}(a_j,\alpha_j) & \underbrace{-\ldots-}_{\text{q-m times}}(b_j,\beta_j)\end{pmatrix}x^{-s}\ \text{(*For FoxH*)}$$

$$\frac{\prod_{j=1}^{m}\text{Gamma}[b_j+s]\ \prod_{j=1}^{n}\text{Gamma}[1-a_j-s]}{\prod_{j=n+1}^{p}\text{Gamma}[a_j+s]\ \prod_{j=m+1}^{q}\text{Gamma}[1-b_j-s]}\,x^{-s}\leftrightarrow$$

$$\begin{pmatrix}\overbrace{+\ldots+}^{\text{m times}}b_j & \overbrace{-\ldots-}^{\text{n times}}a_j\\ \underbrace{+\ldots+}_{\text{p-n times}}a_j & \underbrace{-\ldots-}_{\text{q-m times}}b_j\end{pmatrix}x^{-s}\ \text{(*For MeijerG*)}$$

$$\frac{\text{Gamma}[b_1+\beta_1 s]}{1}\,x^{-s}\leftrightarrow H^{1,0}_{0,1}\left[z\,\middle|\,\begin{array}{c}\cdot\\(b_j,\beta_j)_{1,1}\end{array}\right]x^{-s}\leftrightarrow\begin{pmatrix}+\beta_1 & \cdot\\ \cdot & \cdot\end{pmatrix}x^{-s}\ \text{(*Laplace*)}$$

$$\frac{\text{Gamma}[b_1+\beta_1 s]}{\text{Gamma}[a_1+\alpha_1 s]}\,x^{-s}\leftrightarrow$$

$$H^{1,0}_{1,1}\left[z\,\middle|\,\begin{matrix}(a_i,\alpha_i)_{1,1}\\(b_j,\beta_j)_{1,1}\end{matrix}\right]x^{-s}\leftrightarrow\begin{pmatrix}+\beta_1 & \cdot\\+\alpha_1 & \cdot\end{pmatrix}x^{-s}\quad \text{(* Hilbert, Riesz,2 Hankel, Weyl \& Riemann *)}$$

$$\frac{\text{Gamma}[b_1+\beta_1 s]\ \text{Gamma}[1-a_1-\alpha_1 s]}{\text{Gamma}[a_2+\alpha_2 s]\ \text{Gamma}[1-b_2-\beta_2 s]}\,x^{-s}\leftrightarrow$$

$$H^{1,1}_{2,2}\left[z\,\middle|\,\begin{matrix}(a_i,\alpha_i)_{1,2}\\(b_j,\beta_j)_{1,2}\end{matrix}\right]x^{-s}\leftrightarrow\begin{pmatrix}+\beta_1 & -\alpha_1\\+\alpha_2 & -\beta_1\end{pmatrix}x^{-s}\quad \text{(* Hilbert, Riesz,2 Hankel, Weyl \& Riemann *)}$$

FractionalOrderD case (Riemann-Liouville integral)

In[•]:=

$$\text{With}\Big[\{m=1,n=0,p=1,q=1\},\ \Big\{\text{FoxH}[\{\text{Table}[\{a_i,\alpha_i\},\{i,1,n\}],\text{Table}[\{a_i,\alpha_i\},\{i,n+1,p\}]\},\ \{\text{Table}[\{b_i,\beta_i\},\{i,1,m\}],\text{Table}[\{b_i,\beta_i\},\{i,m+1,q\}]\},\ z],\ \frac{1}{2\pi i}\text{ContourIntegrate}\Big[\frac{\left(\prod_{j=1}^{m}\text{Gamma}[b_j+\beta_j s]\right)\prod_{i=1}^{n}\text{Gamma}[1-a_i-\alpha_i s]}{\left(\prod_{i=n+1}^{p}\text{Gamma}[a_i+\alpha_i s]\right)\prod_{j=m+1}^{q}\text{Gamma}[1-b_j-\beta_j s]}z^{-s},\{s,\mathcal{L}\}\Big]\Big\}\Big]$$

Out[•]=

$$\left\{\text{FoxH}[\{\{\},\{\{a_1,\alpha_1\}\}\},\{\{\{b_1,\beta_1\}\},\{\}\},z],\ -\frac{i\ \text{ContourIntegrate}\left[\frac{z^{-s}\,\text{Gamma}[b_1+s\,\beta_1]}{\text{Gamma}[a_1+s\,\alpha_1]},\{s,\mathcal{L}\}\right]}{2\pi}\right\}$$

In[•]:=

$$\text{Residue}\left[\frac{z^{-s}\,\text{Gamma}[b_1+s\,\beta_1]}{\text{Gamma}[a_1+s\,\alpha_1]},\left\{s,\frac{-k-b_1}{\beta_1}\right\},\ \text{Assumptions}\to\{k\in\text{Integers}\,\&\&\,k\geq 0\,\&\&\,\beta_1\in\text{Reals}\,\&\&\,\beta_1>0\}\right]$$

Out[•]=

$$\frac{(-1)^k\,z^{\frac{k}{\beta_1}+\frac{b_1}{\beta_1}}}{k!\ \text{Gamma}\left[a_1-\frac{(k+b_1)\,\alpha_1}{\beta_1}\right]\beta_1}$$

$$\text{FoxH}[\{\{\},\{\{a_1,\alpha_1\}\}\},\{\{\{b_1,\beta_1\}\},\{\}\},z] == \frac{1}{2\pi i}\text{ContourIntegrate}\left[\frac{z^{-s}\,\text{Gamma}[b_1+s\,\beta_1]}{\text{Gamma}[a_1+s\,\alpha_1]},\{s,\mathcal{L}\}\right] ==$$

$$\text{Sum}\left[\frac{(-1)^k z^{\frac{k}{\beta_1}+\frac{b_1}{\beta_1}}}{k!\,\text{Gamma}\left[a_1-\frac{(k+b_1)\,\alpha_1}{\beta_1}\right]\beta_1},\{k,0,\infty\}\right]$$

In particular, for $\beta_1 == \alpha_1$ we have:

$$\text{FoxH}[\{\{\},\{\{a_1,\alpha_1\}\}\},\{\{\{b_1,\alpha_1\}\},\{\}\},z] ==$$

In[∘]:=

$$\text{Sum}\left[\frac{(-1)^k z^{\frac{k}{\beta_1}+\frac{b_1}{\beta_1}}}{k!\,\text{Gamma}\left[a_1-\frac{(k+b_1)\,\alpha_1}{\beta_1}\right]\beta_1},\{k,0,\infty\}\right]\ /.\ \beta_1\to\alpha_1\ //\ \text{Simplify}$$

Out[∘]=

$$\frac{z^{\frac{b_1}{\alpha_1}}\left(1-z^{\frac{1}{\alpha_1}}\right)^{-1+a_1-b_1}}{\text{Gamma}[a_1-b_1]\,\alpha_1}$$

In[∘]:=

$$\left\{\text{FoxH}[\{\{\},\{\{a_1,\alpha_1\}\}\},\{\{\{b_1,\alpha_1\}\},\{\}\},z],\frac{z^{\frac{b_1}{\alpha_1}}\left(1-z^{\frac{1}{\alpha_1}}\right)^{-1+a_1-b_1}}{\text{Gamma}[a_1-b_1]\,\alpha_1}\right\}\ /.$$
$$\left\{z\to\frac{\text{Random}[\,]}{10},a_1\to\text{Random}[\,],\alpha_1\to\text{Random}[\,],b_1\to\text{Random}[\,]\right\}$$

Out[∘]=

{0.0825744, 0.0825744}

In[∘]:=

$$\{\text{FoxH}[\{\{\},\{\{a_1,\alpha_1\}\}\},\{\{\{b_1,\alpha_1\}\},\{\}\},z],0\}\ /.$$
$$\{z\to\text{Random}[\,]+10,a_1\to\text{Random}[\,],\alpha_1\to\text{Random}[\,],b_1\to\text{Random}[\,]\}$$

Out[∘]=

{0., 0}

$$\text{FoxH}[\{\{\},\{\{a_1,\alpha_1\}\}\},\{\{\{b_1,\alpha_1\}\},\{\}\},z] == \begin{cases}\frac{z^{\frac{b_1}{\alpha_1}}\left(1-z^{\frac{1}{\alpha_1}}\right)^{-1+a_1-b_1}}{\text{Gamma}[a_1-b_1]\,\alpha_1} & \text{Abs}[z]^{\frac{1}{\alpha_1}}<1\\ 0 & \text{True}\end{cases}$$

Meijer kernel BesselK case

In[•]:=

$$\text{With}\Big[\{m = 2, n = 0, p = 0, q = 2\},\ \Big\{\text{FoxH}[\{\text{Table}[\{a_i, \alpha_i\}, \{i, 1, n\}], \text{Table}[\{a_i, \alpha_i\}, \{i, n+1, p\}]\}, \{\text{Table}[\{b_i, \beta_i\}, \{i, 1, m\}], \text{Table}[\{b_i, \beta_i\}, \{i, m+1, q\}]\}, z],\ \frac{1}{2\pi i}\text{ContourIntegrate}\Big[\frac{\left(\prod_{j=1}^{m}\text{Gamma}[b_j + \beta_j s]\right)\prod_{i=1}^{n}\text{Gamma}[1 - a_i - \alpha_i s]}{\left(\prod_{i=n+1}^{p}\text{Gamma}[a_i + \alpha_i s]\right)\prod_{j=m+1}^{q}\text{Gamma}[1 - b_j - \beta_j s]} z^{-s}, \{s, \mathcal{L}\}\Big]\Big\}\Big]$$

Out[•]=

$$\Big\{\text{FoxH}[\{\{\}, \{\}\}, \{\{\{b_1, \beta_1\}, \{b_2, \beta_2\}\}, \{\}\}, z],\ -\frac{i\ \text{ContourIntegrate}[z^{-s}\,\text{Gamma}[b_1 + s\,\beta_1]\ \text{Gamma}[b_2 + s\,\beta_2], \{s, \mathcal{L}\}]}{2\pi}\Big\}$$

In[•]:=

$$\text{Sum}\Big[\text{Residue}\Big[z^{-s}\,\text{Gamma}[b_1 + s\,\beta_1]\ \text{Gamma}[b_2 + s\,\beta_2], \Big\{s, \frac{-k - b_1}{\beta_1}\Big\}, \text{Assumptions} \to \{k \in \text{Integers} \&\& k \geq 0 \&\& \beta_1 \in \text{Reals} \&\& \beta_1 > 0\}\Big], \{k, 0, \infty\}\Big] + \text{Sum}\Big[\text{Residue}\Big[z^{-s}\,\text{Gamma}[b_1 + s\,\beta_1]\ \text{Gamma}[b_2 + s\,\beta_2], \Big\{s, \frac{-k - b_2}{\beta_2}\Big\}, \text{Assumptions} \to \{k \in \text{Integers} \&\& k \geq 0 \&\& \beta_2 \in \text{Reals} \&\& \beta_2 > 0\}\Big], \{k, 0, \infty\}\Big]$$

Out[•]=

$$\sum_{k=0}^{\infty}\frac{(-1)^k z^{\frac{k}{\beta_1}+\frac{b_1}{\beta_1}}\,\text{Gamma}\Big[b_2 - \frac{(k+b_1)\,\beta_2}{\beta_1}\Big]}{k!\ \beta_1} + \sum_{k=0}^{\infty}\frac{(-1)^k z^{\frac{k}{\beta_2}+\frac{b_2}{\beta_2}}\,\text{Gamma}\Big[b_1 - \frac{(k+b_2)\,\beta_1}{\beta_2}\Big]}{k!\ \beta_2}$$

Case $\beta_1 == \beta_2$:

In[∘]:=

$$\sum_{k=0}^{\infty}\frac{(-1)^k z^{\frac{k}{\beta_1}+\frac{b_1}{\beta_1}}\,\text{Gamma}\left[b_2-\frac{(k+b_1)\,\beta_2}{\beta_1}\right]}{k!\,\beta_1}+\sum_{k=0}^{\infty}\frac{(-1)^k z^{\frac{k}{\beta_2}+\frac{b_2}{\beta_2}}\,\text{Gamma}\left[b_1-\frac{(k+b_2)\,\beta_1}{\beta_2}\right]}{k!\,\beta_2}\ /.\ \{\beta_2\to\beta_1\}$$

Out[∘]=

$$\frac{z^{\frac{b_1}{\beta_1}}\left(z^{\frac{1}{2\beta_1}}\right)^{-b_1+b_2}\text{BesselI}\left[b_1-b_2,\,2\,z^{\frac{1}{2\beta_1}}\right]\text{Gamma}[1+b_1-b_2]\ \text{Gamma}[-b_1+b_2]}{\beta_1}+\frac{z^{\frac{b_2}{\beta_1}}\left(z^{\frac{1}{2\beta_1}}\right)^{b_1-b_2}\text{BesselI}\left[-b_1+b_2,\,2\,z^{\frac{1}{2\beta_1}}\right]\text{Gamma}[b_1-b_2]\ \text{Gamma}[1-b_1+b_2]}{\beta_1}$$

In[∘]:=

```
% // PowerExpand // FullSimplify
```

Out[∘]=

$$\frac{2\,z^{\frac{b_1+b_2}{2\beta_1}}\ \text{BesselK}\left[b_1-b_2,\,2\,z^{\frac{1}{2\beta_1}}\right]}{\beta_1}$$

In[∘]:=

$$\left\{\text{FoxH}[\{\{\},\{\}\},\{\{\{b_1,\beta_1\},\{b_2,\beta_2\}\},\{\}\},z],\ \frac{2\,z^{\frac{b_1+b_2}{2\beta_1}}\ \text{BesselK}\left[b_1-b_2,\,2\,z^{\frac{1}{2\beta_1}}\right]}{\beta_1}\right\}\ /.\ \{\beta_2\to\beta_1\}\ /.\left\{z\to\frac{\text{Random}[\,]}{10},\ a_1\to\text{Random}[\,],\ b_2\to\text{Random}[\,],\ \beta_1\to\text{Random}[\,],\ b_1\to\text{Random}[\,]\right\}$$

Out[∘]=

{0.534003, 0.534003}

$$\text{FoxH}[\{\{\},\{\}\},\{\{\{b_1,\beta_1\},\{b_2,\beta_1\}\},\{\}\},z]==\frac{2\,z^{\frac{b_1+b_2}{2\beta_1}}\ \text{BesselK}\left[b_1-b_2,\,2\,z^{\frac{1}{2\beta_1}}\right]}{\beta_1}$$

$$\text{Logarithmicalcase},\ \ \frac{-k_1-b_1}{\beta_1}==\frac{-k_2-b_2}{\beta_2}:$$

In[∘]:=

$$\text{Solve}\left[\frac{-k_1-b_1}{\beta_1}-\frac{-k_2-b_2}{\beta_2}==\epsilon,k_2\right]\ //\ \text{ExpandAll}$$

Out[∘]=

$$\left\{\left\{k_2\to -b_2+\epsilon\,\beta_2+\frac{b_1\,\beta_2}{\beta_1}+\frac{k_1\,\beta_2}{\beta_1}\right\}\right\}$$

In[∘]:=

$$\left\{z^{-s}\,\text{Gamma}[b_1+s\,\beta_1]\,\text{Gamma}[b_2+s\,\beta_2],\frac{-k_1-b_1}{\beta_1},\frac{-k_2-b_2}{\beta_2}\right\}\ /.\ \{\beta_1\to 2,\beta_2\to 3\}$$

Out[∘]=

$$\left\{z^{-s}\,\text{Gamma}[2\,s+b_1]\,\text{Gamma}[3\,s+b_2],\frac{1}{2}\,(-b_1-k_1),\frac{1}{3}\,(-b_2-k_2)\right\}$$

In[∘]:=

$$\text{Solve}\left[\frac{1}{2}\,(-b_1-k_1)-\frac{1}{3}\,(-b_2-k_2)==0,k_2\right]$$

Out[∘]=

$$\left\{\left\{k_2\to\frac{1}{2}\,(3\,b_1-2\,b_2+3\,k_1)\right\}\right\}$$

$$\text{Residue}\left[z^{-s}\,\text{Gamma}[2\,s+b_1]\,\text{Gamma}[3\,s+b_2],\left\{s,\frac{-k_1-b_1}{\beta_1}\right\}\right]$$

In[∘]:=

$$z^{-s}\,\text{Gamma}[b_1+s\,\beta_1]\,\text{Gamma}[b_2+s\,\beta_2]\ /.\left\{s\to\frac{-k_2-b_2}{\beta_2}+\epsilon\right\}\ /.$$
$$\left\{k_2\to -b_2+4\,\epsilon\,\beta_2+\frac{b_1\,\beta_2}{\beta_1}+\frac{k_1\,\beta_2}{\beta_1}\right\}\ //\ \text{ExpandAll}$$

Out[∘]=

$$z^{3\,\epsilon+\frac{b_1}{\beta_1}+\frac{k_1}{\beta_1}}\,\text{Gamma}[-k_1-3\,\epsilon\,\beta_1]\,\text{Gamma}\left[b_2-3\,\epsilon\,\beta_2-\frac{b_1\,\beta_2}{\beta_1}-\frac{k_1\,\beta_2}{\beta_1}\right]$$

In[∘]:=

```
Series[ (Series[Gamma[-k1 - 3 ϵ β1], {ϵ, 0, 1}, Assumptions→
        {k1 ∈ Integers&& k1 ≥ 0 && k2 ∈ Integers&& k2 ≥ 0}] // Normal)
   (Series[z^-((-k2-b2)/β2 + ϵ) Gamma[-k2 - 3 ϵ β2], {ϵ, 0, 1},
        Assumptions→ {k1 ∈ Integers&& k1 ≥ 0 && k2 ∈ Integers&& k2 ≥ 0}] // Normal),
  {ϵ, 0, -1}, Assumptions→ {k1 ∈ Integers&& k1 ≥ 0 && k2 ∈ Integers&& k2 ≥ 0}]
```

Out[∘]=

$$\frac{(-1)^{k_1+k_2} z^{\frac{b_2+k_2}{\beta_2}}}{9\, k_1!\, k_2!\, \beta_1 \beta_2 \epsilon^2} - \frac{(-1)^{k_1+k_2} z^{\frac{b_2}{\beta_2}+\frac{k_2}{\beta_2}} (\text{Log}[z] + 3\,\text{PolyGamma}[0, 1+k_1]\, \beta_1 + 3\,\text{PolyGamma}[0, 1+k_2]\, \beta_2)}{9\, (k_1!\, k_2!\, \beta_1 \beta_2)\, \epsilon} + O[\epsilon]^0$$

In[∘]:=

```
Series[Gamma[-k2 - 3 ϵ β2], {ϵ, 0, 1},
   Assumptions→ {k1 ∈ Integers&& k1 ≥ 0 && k2 ∈ Integers&& k2 ≥ 0}] // Normal
```

Out[∘]=

$$\frac{(-1)^{k_2}\, \text{PolyGamma}[0, 1+k_2]}{k_2!} - \frac{(-1)^{k_2}}{3\, \epsilon\, k_2!\, \beta_2} - \frac{(-1)^{k_2}\, \epsilon \left(\pi^2 + 3\,\text{PolyGamma}[0, 1+k_2]^2 - 3\,\text{PolyGamma}[1, 1+k_2]\right) \beta_2}{2\, k_2!}$$

In[∘]:=

```
Series[z^(3 ϵ + b1/β1 + k1/β1) Gamma[-k1 - 3 ϵ β1] Gamma[-k2 - 3 ϵ β2], {ϵ, 0, -1},
   Assumptions→ {k1 ∈ Integers&& k1 ≥ 0 && k2 ∈ Integers&& k2 ≥ 0}] // Normal
```

Out[∘]=

$$\frac{(-1)^{k_1+k_2} z^{\frac{b_1+k_1}{\beta_1}}}{9\, \epsilon^2\, k_1!\, k_2!\, \beta_1 \beta_2} - \frac{(-1)^{k_1+k_2} z^{\frac{b_1}{\beta_1}+\frac{k_1}{\beta_1}} (-\text{Log}[z] + \text{PolyGamma}[0, 1+k_1]\, \beta_1 + \text{PolyGamma}[0, 1+k_2]\, \beta_2)}{3\, \epsilon\, k_1!\, k_2!\, \beta_1 \beta_2}$$

In[∘]:=

```
Residue[z^(ϵ + b1/β1 + k1/β1) Gamma[-k1 - ϵ β1] Gamma[b2 - ((b1 + k1 + 2 ϵ β1) β2)/β1],
  {s, (-k1 - b1)/β1}, Assumptions→ {k1 ∈ Integers&& k1 ≥ 0 && β1 ∈ Reals&& β1 > 0}]
```

Out[∘]=

0

```
z^(ϵ + b1/β1 + k1/β1) Gamma[-k1 - ϵ β1] Gamma[-k2 - ϵ β2]
```

In[∘]:=

```
Sum[Residue[z^-s Gamma[b1 + s β1] Gamma[b2 + s β2], {s, (-k - b1)/β1},
    Assumptions→ {k ∈ Integers&& k ≥ 0 && β1 ∈ Reals&& β1 > 0}], {k, 0, ∞}] +
  Sum[Residue[z^-s Gamma[b1 + s β1] Gamma[b2 + s β2], {s, (-k - b2)/β2},
    Assumptions→ {k ∈ Integers&& k ≥ 0 && β2 ∈ Reals&& β2 > 0}], {k, 0, ∞}]

Series[z^-s Gamma[b1 + s β1] Gamma[b2 + s β2], {k, 0, ∞}]
```